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Task Design for Klein's Second Discontinuity

Desenvolvimento de Tarefa para a Dupla Descontinuidade de Klein

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ABSTRACT

Since the 19th century, studies of mathematics at university have been a main component of the usual preparation for teaching at secondary level. Already around 1900, Klein pointed out that specific measures are needed to ensure that the university mathematical preparation becomes useful to the teacher, and he insisted that universities themselves must take responsibility for these measures. In this paper, we discuss this problem, as it presents itself in 2022, and we present and exemplify some principles of task design which are intended to support students' mobilisation of university mathematical knowledge in relation to specific mathematical challenges for high school teachers[3].

KEYWORDS: Klein's Second Discontinuity. Task Design. Anthropological Theory of the Didactic.

RESUMO

Desde o século XIX, os estudos de Matemática nos currículos universitários têm sido uma componente principal da preparação para o ensino dessa disciplina no Ensino Médio. Já por volta de 1900, Felix Klein destacou que, para garantir que a preparação matemática na universidade possa ser útil para o futuro professor, são necessárias ações específicas. Nesse sentido, ele reforçou que as próprias universidades devem assumir a responsabilidade por essas medidas. Neste artigo, discutimos esse problema e mostramos como ele se apresenta em 2022. Além disso, apresentamos e exemplificamos alguns princípios do desenvolvimento de tarefas que se destinam a apoiar a mobilização pelos estudantes dos conhecimentos matemáticos acadêmicos em relação aos desafios matemáticos específicos para professores do Ensino Médio.

PALAVRAS-CHAVE: Dupla Descontinuidade de Klein. Desenvolvimento de Tarefa. Teoria Antropológica do Didático.

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The Problem

The gap between university mathematics and secondary school mathematics has widened considerably in the 20th century. At the same time, the two institutions and their mathematical disciplines are not monolithic. At universities, mathematics has developed into several related, yet quite different disciplines of teaching and research, including the various domains of “pure mathematics” but also other disciplinary branchings referred to by labels such as applied mathematics, data science, computer science, statistics and even parts of engineering, finance and other profession oriented sciences. All of these are to some extent “references” for secondary school level mathematics: particularly at the upper secondary level (with students aged 15-16 onwards), various “streams” are present in most countries, which not only offer more or less mathematics, but also mathematics which can be more or less closely related to the university level forms of mathematics. This complicates both of the transition problems, described by Klein as the “double discontinuity” (KLEIN, 1908/2016; WINSLØW; GRØNBÆK, 2014). One cannot simply talk of one “school mathematics” and of one “university mathematics”.

Here we will consider only the second discontinuity, between university studies of mathematics (in some form) and teaching secondary level mathematics (in some form). It arises for university students as they prepare to become secondary level teachers. In most countries, this involves some mixture of university mathematics studies (some are in fact designed for teachers while some are not) where in general the second discontinuity must be considered. We do not consider generic educational components here, but only those parts that are directly aimed at adapting the future teachers’ mathematical knowledge to presumed needs for teaching at the secondary level, with its variation and the rest of the students’ university mathematical preparation in mind.

Even in European countries, the organization of teacher education programmes – and in particular, the part we focus on here – varies considerably. It can be considered as a bridging problem, where the continents to be bridged are two curricula: the mandatory programme of mathematics units studied at university (excluding the profession oriented part), and the secondary curricula in which the student will be teaching. In many countries (such as France and Denmark), there is a “consecutive” organization where the professional part comes last; in other countries, like Germany, a more “parallel” organization can be found.

We will now further delimit our problem, considering with Watson and Ohtani (2015, p. 3) that “the detail and content of tasks have a significant effect on learning; from a cultural perspective, tasks shape the learners’ experience of the subject and their understanding of the nature of mathematical activity; from a practical perspective, tasks are the bedrock of classroom life”. This leads us to the following general problem: *What are possible principles for designing tasks which, within a consecutive model, allow future teachers to adapt their mathematical background to the professional tasks of teaching mathematics at secondary level? In particular, how do these principles relate to mathematical tasks worked on by the future teachers at university, and by students in the schools where they prepares to teach?*

We note that similar questions were studied by Bauer (2013) within the parallel model found in Germany. Naturally, the answers proposed here cannot exhaust the full range of relevant principles for task design related to courses in the consecutive model, but the focus on “adapting their mathematical background” will nevertheless allow us to propose a reasonably complete set of principles. Our discussion, at the end of the paper, will focus on the extent to which the proposed principles may be adapted or even extended to other similar, but different contexts. However, even before that, we need to furnish a more precise framework for the above problem, and then present our context, principles, and some examples of their use.

Theoretical Framework and Research Question

We adopt, from the anthropological theory of the didactic (ATD), the notion of *institution*, which, are roughly speaking, social systems. This wide and unclear definition can be made more precise (see e.g., CHEVALLARD, 2019, p. 92): human beings occupy, throughout their lives, various positions p in different institutions I, and for each of these positions, certain relationships, denoted $R_I(p,O)$ are required to certain objects O (O can be, for instance, knowledge objects, physical entities etc.). One can even define the elusive notion of institution as being a configuration of positions, each defined by a set of such relationships which occupants of the position is required to have in order to occupy p within I. For instance, to be a teacher t in a school institution S, it is required to hold certain relationships to a number of didactical tasks, to hold certain degrees etc. – these tasks, degrees and so on all being objects whose existence for S is confirmed by the relationships held by (some) positions in S.

Institutions may come in types such as schools and universities. Institutions may also, at least apparently “share” objects, for instance in the sense that they label certain objects in the same way. Yet, what is labelled, for instance, “real numbers” and "algebra" may not only differ from institution to institution, but even from position to position within these.

The second discontinuity has been modelled, within this framework, by Winsløw (2013) as pertaining to passages of the type

$$R_U(\sigma,\omega) \to R_S(t\,,O)$$

where U is the university institution, σ is a student in U, ω is a (mathematical) knowledge object to which σ is required to hold the relationship $R_U(\sigma,\omega)$; and S is a school institution, t is a mathematics teacher in S, and O is an object to which t is required to hold the relationship $R_S(t,O)$. For the passage to be meaningful, it is naturally expected that $R_U(\sigma,\omega)$ is of some relevance to $R_S(t\,,O)$, so that the latter could be supported by the former, presumably with some further development. If this development occurs, at least in part, already within the university institution, we can rewrite the above passage as

$$R_U(\sigma,\omega) \to R_U(\sigma,O) \cong R_S(t,O)$$

where $R_U(\sigma\,,O) \cong R_S(t,O)$ indicates an approximate similarity of the relationship obtained by σ within U and the relationship to be held by t within S. If working with a task of type T can achieve the passage $R_U(\sigma,\omega) \to R_U(\sigma,O)$, at least in part, we write

$$R_U(\sigma,\omega) \xrightarrow{T} R_U(\sigma,O)$$

In this paper, we now consider the following research question: Given a relationship $R_S(t,O)$ required to occupy t in S, how could some T be designed so that σ could develop $R_U(\sigma\,,O)$, with $R_U(\sigma,O) \cong R_S(t,O)$, based on some $R_U(\sigma,\omega)$? In other words, what principles can be formulated for the design of T?

Here, we present and explore four principles which have progressively been identified in the course of more than a decade of task design in the context described in the next section. The principles each focus on O at one of the praxeological levels (type of task, technique, technology and theory – for definitions of these ATD notions, see e.g., CHEVALLARD, 2019, p. 91-92).

P1. In case O is a *mathematical type of task taught in S*, T is simply a task of which is somewhat more demanding – but otherwise similar – to O, with the additional demands being satisfied by drawing on some $R_U(\sigma,\omega)$.

P2. With O as in P1, T requires σ to *pose* a task of type O, based on some $R_U(\sigma,\omega)$ which may also lead to a more theoretical or structured relation $R_U(\sigma,O)$.

P3. If O is *one or more mathematical techniques* (authentic or imaginary, correct or. erroneous), which t should be able to foresee and assess, then T could ask σ to foresee or assess O while drawing on some ω.

P4. If O is *a segment of mathematical technology or theory*, which t should teach or. otherwise know, then T could demand that σ establishes whether O is mathematically consistent with ω – for instance, can be proved based on ω.

Klein's discontinuities focus on the future teachers' relation to (school) *mathematical* objects. Klein points out that it is potentially useful for future teachers to establish such relations $R_U(\sigma,O)$ from the "higher standpoint" of university mathematics (an element of which we denote here by ω); but that doing so requires deliberate support measures within U, here conceived as engaging σ in work with carefully designed tasks T. The above principles then distinguish, but do not exhaust, important cases for the construction of T, as we will show through examples.

To prevent misconceptions, we also underline that the research question – and therefore the list – does not pretend to cover all task design that may be relevant to mathematics teacher education. Indeed, future teachers also need to develop didactical knowledge (both practical and theoretical) that cannot be directly supported by elements of "pure" mathematics as learned in standard university courses.

We will now outline the concrete context in which the four principles have emerged, and then present and analyse some examples of concrete T designed with them.

Context

The principles P1-P4 have progressively been made explicit in the theoretical terms given above, as they were developed and used within a concrete context by the first author (since about 2009). This context is a course, called "Mathematics for the teaching context" (UvMat), offered at the University of Copenhagen to students who do a minor in mathematics in view of becoming high school teachers.

We now outline what $R_U(\sigma,\omega)$, and in particular ω, could be in this context (for minor students σ). Before UvMat, σ has taken at least 1.25 years' credit of mathematics courses, covering: one- and multi-variable calculus, linear algebra (including axiomatic vector space theory), ordinary differential equations, abstract algebra (rings, fields and groups), differential geometry, discrete mathematics, statistics and probability, and analysis up to Fourier and metric space theory. The calculus part involves some level of computer algebra use.

We note that the mathematics courses drawn on are basic courses in the bachelor programme on pure mathematics. They focus primarily on theory development, and students are required to solve relatively theoretical tasks (except for the calculus part) involving deductive reasoning. Virtually no examples are studied of how the theory applies to solve practical problems outside of pure mathematics.

What, by contrast, could $R_S(t,O)$ – and in particular O – be? Danish high school mathematics has several levels and variations, but the core could be described as a study of concrete one variable functions and models based on such, up to practical uses of differential and integral calculus. Other mandatory domains are probability, statistics and geometry. Students' grades in mathematics depend largely on their ability to solve standard tasks, with or without the use of computer algebra systems. Deductive reasoning still appears, but recent curricula give reinforced attention to modelling and interactions with other high school disciplines, and to mathematical inquiry. Also, the use of computer tools – especially computer algebra systems – is strongly emphasised.

A bridge needs solid bases on both sides. It is not an aim for UvMat to connect all of the mathematical background of students to all of the high school mathematics they will have to deal with as teachers, but each task in UvMat must have solid connections to *both*, and in particular focus on important aspects of high school mathematics.

Examples and Analysis ff Tasks Designed from P1-P4

In the following, the tasks we present come from the final exam in the course; former exam items (from an inventory of well over 100) are also used as exercises in the course. The students are informed, though, that exam items are never mere variations of former exam items; they always require the student to create new connections between course contents and university mathematics, and the high school object involved. Thus, the continuous development of such tasks form a central challenge of running UvMat. As exam tasks need to be relatively simple, the course also involves more involved assignments (see HUO, 2023, for an in-depth analysis of an example).

P1: Solving "Advanced Variations" of School Mathematical Tasks

Mathematics teachers naturally need to be able solve the tasks given to students, and some of the items worked on in UvMat are merely advanced variants of high school tasks (often involving non-trivial construction of a mathematical model, e.g. for a probability item). Here is an example:

Britta participates in a multiple choice test with *n* questions. For every question, one can choose among 3 possible answers, of which only one is correct. Passing the test requires that one chooses correct answers for at least half of the questions. Britta knows nothing of the subject and answers randomly.

a) If there are 10 questions, what is the probability that Britta passes the exam?

b) If the test is to be made, so that students like Britta has less than a 5% chance to pass, how big must *n* then be? Explain your answer.

(Exam June 2019, exercise 5)

Question a) is a standard application of the binomial distribution and, as such, is simply a high school level task. It merely prepares the second question b), which is technically harder, as the parameter *n* is unknown, rather than given. Students use tools to compute values of the binomial distribution function corresponding to given values of the parameters *n* and *p* (the latter being $\frac{1}{3}$ here). Another difficulty is that the meaning of "at least half" depends on whether n is even or odd. Many students will solve b) by computing, for increasing values of n, the binomial distribution function $F_{n,1/3}$ at something like $n/2$. Since $F_{n,1/3}$ is only defined on $\{0,1,\dots,n\}$, many students simply look for the first even *n* where $F_{n,1/3}\,(n/2) > 0.95$, which in this case is 30. However, the correct answer is, in fact, just 23 (we leave it to the readers to work out the details, using some software able to do compute binomial distributions).

To understand the kinds of ω elements to be drawn on here, we note that b) is quite similar to "inverse problems" related to probability distributions. The students have indeed encountered such problems in connection with subjects like confidence intervals, treated both in this course (from a high school perspective) and in the predecessor courses on statistics (at a technically more advanced level). Thus, solving b) certainly draws on very specific theoretical and technical elements ω, in addition to familiarity with a certain computer algebra system (Maple), which they have developed both in UvMat and in some of the prerequisite bachelor courses. This also illustrates that a variety of $R_U(\sigma,\omega)$ – even with ω being rather far from the school mathematical type of task *O* – could be relevant to such "extended" tasks of type *O*.

P2: Posing School Mathematical Tasks

Some of the mathematical tasks, which teachers need to solve regularly, are related to *preparing tasks for their students* – either by selection or construction. Here, more advanced mathematical work than merely solving tasks can be involved. The following UvMat item is an attempt to generate such a more advanced perspective:

We say that a quadratic equation is *nice* if it is of form $x^2 + bx + c = 0$ where b and c are integers.

a) If you are given two integers m and n, how can you construct a nice quadratic equation with m and n as solutions? Explain a method and give an example of how it works.

b) Can you construct a nice quadratic equation that has both a rational and an irrational solution? Explain.

(Exam August 2019, exercise 2)

The mathematical elements from university mathematics which need to be drawn on here, can be summarized as: knowing how to use a formal, *ad hoc* definition ("nice quadratic") without confounding the definition with an everyday conception of "nice"; a result about polynomials ("a is a root of $p(x)$ if and only if $x - a$ is a factor in $p(x)$"; and some experience with reasoning about irrational numbers. The latter may sound a bit vague, but in fact, b) can be solved in many ways – the most complicated probably being to use the quadratic formula. A better way is to use an observation easily made from a), namely that the product of the solutions is c, while this product will be irrational in the case described in b). Thus, very simple facts about polynomials and irrational numbers – not currently taught in high school, but certainly encountered at university – are activated here, to address what is clearly a relevant mathematical task for high school teachers, given that quadratic equations are taught and used there. We note that the kind of $R_U(\sigma,O)$ built here is typically more theoretical and less technical than what is aimed at in P1, corresponding to mobilizing more theoretical parts of ω.

P3: Assessing or Imagining Student Techniques

Logarithm functions and their use form part of the core content in high school mathematics. Part of the difficulty is the "indirect" definition they are usually given, as inverse functions of exponential functions (cf. also P4 below). On the other hand, this theoretical definition is important in many frequent practices, such as solving equations involving exponents. The following item directly attacks such situation:

Peter and Lise have to decide whether the equation $\ln y^2 = e^{-x}$ defines a function (with y as a function of x). Peter says: "Yes, for the equation can be rewritten as $2\ln y = e^{-x}$, so $y = \exp\left(\frac{1}{2}e^{-x}\right)$." Lise says: "No, for the equation can be rewritten as $y^2 = \exp\left(e^{-x}\right)$, so for each x-value there are two y-values".

a) Who is right? Give a detailed explanation of the correct answer.

b) One of the two answers is false. Explain where the error arises.

(Exam June 2014, exercise 3)

We first note that this exercise involves a school mathematical task ("does the equation $\ln y^2 = e^{-x}$ define a function…") but the tasks given to the university students in the above item is at another level: consider some (imaginary) student solutions, decide whether they are correct, and explain why. In fact, it is a crucial teacher task to relate to students' mathematical work and provide feedback; items formed in this manner are therefore found occasionally (such in about 1 in 10 of the exercises proposed) throughout the course. As for the mathematical contents, students will know the identity $\ln x^a = a\ln x$ but may not have thought about that it is only valid, and meaningful, for positive values of x. In the exercise, Peter makes a mistake in his first "rewriting" of the equation, since the given equation is also meaningful for negative y, while the second is not.

The university knowledge, which the students could apply to solve this item, is not very advanced. They have certainly worked with more formal (set-theoretical) definitions of functions than what is seen in high school, but the informal definition ("to each value of x there must correspond exactly one y") suffices to realize that Lise is right, and then look for an error in Peter's rewritings. That identities such as $\ln x^a = a\ln x$ may have restricted validity (beyond what makes the expressions meaningful) is also something which university studies could increase students' awareness of. In particular, unlike in high school, they would often see qualifications like "for all $a \in \mathbb{R}, x > 0$", following an identity. So we can say that, in addition to the explicit treatment (in the UvMat course) of logarithms, the theoretical notion of function, and the logical subtleties related to equation solving (in particular, implications), the main university mathematical element (ω) to invest in this task is a more developed practice of applying identities only were they are valid. This specific $R_U(\sigma,\omega)$ turns out, in practice, not to be sufficiently developed for many students. Indeed, as observed by Winsløw et al. (2014), working with tasks designed to facilitate some passage $R_U(\sigma,\omega) \rightarrow R_U(\sigma,O)$ often involves "repairing" dysfunctional $R_U(\sigma,\omega)$.

P4: Making New Theoretical Connections

A great deal of the work in UvMat – perhaps as much as half – concerns theoretical aspects of high school mathematics, like proofs of results or constructions which appear more informally in high school, such as the general meaning (and construction) of x^y for $x > 0, y \in \mathbb{R}$ (see WINSLØW; GRØNBÆK, 2014, p. 77-79). The strong focus on theory is in part a consequence of the aim to draw on university elements ω where $R_U(\sigma,\omega)$ is often very limited when it comes to students' practical experience with the praxis level of ω. The general familiarity that students have gained with formal theory is frequently an asset they need to draw on, as they solve tasks based on P4.

Our last example relates to work carried out in the course with the theory behind linear regression. In high school mathematics, it is mainly taught as a practice carried out with some tool like excel, along with informal explanations that this provides "the best linear model" for a given 2d data set. In the course we revisit proofs which the students may have seen in statistics courses, along with more elementary approaches (see WINSLØW; GRØNBÆK, 2014, p. 79-81). The following item links the theoretical problem of "minimizing least squares" to one-dimensional optimization as taught in high school:

A simplified form of linear regression results from requiring that the regression line passes through the point $(0,0)$, so that the equation of the line is of form $y = ax$ where a is a constant.

a) Derive a formula for a corresponding to a data set $(x_1, y_1), \ldots, (x_n, y_n)$, by determining a such that the sum $S(a) = \sum_{k=1}^{n}(y_k - ax_k)^2$ is minimized.

b) Explain how this formula can be used to determine the resistance R in an electrical circuit, based on corresponding values of the current I and the voltage U, knowing that Ohm's law says that $U = RI$.

(Exam January 2012, exercise 5)

It is a fact that optimization of two variable functions is not usually taught in Danish high school, and that alternative approaches to linear regression (as taught in UvMat) are also somewhat beyond what most high school classes would meet, or be able to cope with. While a) and b) rely in principle only on high school mathematics, the theoretical nature of the questions – and the requirements in terms of symbolic computation – certainly draws on $R_U(\sigma,\omega)$ with ω involving both practical and theoretical elements that are not strictly related to linear regression, but nevertheless supposed to be developed or strengthened at university.

The last question is not technically demanding (one should explain translate the formula from a) to give an estimate for R in terms of a set of measurements (I_k, U_k) of current and voltage). Nevertheless, it is important for mathematics teachers at high school to know and integrate models from neighbouring disciplines in their teaching, not least when it comes to statistics topics like the present one. Question b) requires one to make a connection between a simple model from physics and the mathematical result developed in a).

Discussion and Conclusions

To what extent can the principles and examples above be of interest outside of the context in which they arose? To most university teachers and researchers interested in the general problem we described in the introduction, the principles (derived from an ATD model of the problem) would remain arid speculations without the examples. Yet, it is fully possible that many of the same scholars would also not see the relevance of the examples for contexts familiar to them. Indeed, they are more or less arbitrary cases of efforts to link certain objects *O* and ω which occur in Danish high school mathematics and in the first two years of undergraduate mathematics at the University of Copenhagen. Most probably, many of these objects could in themselves be found in equivalent contexts elsewhere, but the emphasis on theoretical and technical aspects of them, reflected in the examples, could still be felt to be less relevant there. For instance, a recent study by Bosch et al. (2021) of external didactical transpositions in university mathematics suggests that in North America, the first years of undergraduate studies are often much more focused on technical aspects of calculus. This might mean that integrating such technical aspects would be seen as much more important than what is reflected by the examples given here, while the emphasis on proof (reflected by P4) would be considered less helpful. Similarly, for contexts in which probability distributions or linear regression do not feature centrally in the secondary curriculum, the example given for P1 and P4 would appear irrelevant.

Thus, to go beyond those examples of *T* – that certainly depend on the specific context – we really need to hold on to *principles* such as P1-P4. They can constitute a framework which could be adapted to such more or less different contexts: engaging students in work with *O* (centrally occurring in secondary mathematics) that is characterised by a focus on different praxeological levels of relevant to future teachers on the one side, and on drawing on similar levels of some central objects ω in students' university mathematical background.

But this conclusion merits other reservations. Applying the four principles take a certain inventory of *O* and ω as given conditions, and the potential as well as the feasibility of linking them as a working hypothesis (following KLEIN, 1908/2016). At least two major questions are left open by this: the question of *external didactical transposition*, both at university and in secondary school, resulting in the given *O* and ω; and the actual *importance of the potential for the subsequent professional practice* of *t* (a position which is somehow put aside by the non-examined "similarity" $R_U(\sigma,O) \cong R_S(t,O)$). The first major question contains in fact two separate aspects: the possibility of inadequacy (or at least needs for development) of secondary mathematics, and the problem of determining the adequate mathematical basis to be developed at university, and the extent to which this should consist in teaching conceived for more general publics than future teachers. The last question has been examined in more detail by Winsløw (to appear), linking it to recent quantitative studies of correlation between teachers' university mathematical background and the quality of their performance as secondary teachers. This kind of research may also have bearings on the second major question, as it involves identifying structural similarities of teacher education programmes that produce, according to such quantitative studies, teachers with high performance (measured by learning gains of the teachers' students, or by measures of teacher knowledge that correlate with high teaching performance). Among these similarities are, in fact, a combination of certain standard undergraduate mathematics units and units focusing on teachers' knowledge of central secondary mathematics. But we still need more direct and theoretically precise ways to investigate how students' participation in such courses affects their later performance as teachers, and in particular how principles for task design may contribute to enhanced effects.

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