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The acceptance of Gödel's Incompleteness Theorem by mathematicians

A acolhida do Teorema da Incompletude de Gödel pelos matemáticos

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ABSTRACT

In this article, one presents how some of the most important mathematicians of the early twentieth century – those responsible for organizing mathematics into algebraic, topological and order structures, which one studied in undergraduate courses in mathematics – received the results established by Gödel´s Incompleteness Theorem. One understands that the stance taken by Bourbaki group and revealed in his works thus claims how the mathematicians of that time welcomed the incompleteness theorem and handled with its consequences. The attitude assumed by the group was to continue doing Mathematics with the same ideal of complete formalization, although they faced with the proof of the existence of a non-empty set of true and undemonstrable propositions – via mathematical tools for producing this science – and of the incompleteness of any theory that contains Peano axioms. One also presents Wittgenstein's perspective on Gödel's Incompleteness Theorem. Finally, one apprehends that this theorem is understood by the mathematical community as a messenger of incompleteness as a characteristic inherent to axiomatization, not as an impediment to the continuation of the activity with formal systems, but as an invigorating result for Mathematics.

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RESUMO

Neste artigo, apresentamos o modo como alguns dos matemáticos mais importantes do início do século XX, aqueles responsáveis pela organização da Matemática em estruturas algébricas, topológicas e de ordem, que estudamos em cursos de graduação em Matemática, acolheram os resultados estabelecidos pelo Teorema da Incompletude de Gödel. Compreendemos que a postura assumida pelo grupo Bourbaki e revelada em suas obras diz desse modo de como os matemáticos da época receberam o teorema da incompletude e conviveram com as consequências dele. A atitude assumida pelo grupo foi seguir fazendo Matemática com o mesmo ideal de formalização completa, embora diante da prova da existência de um conjunto não vazio de proposições verdadeiras e indemonstráveis - via ferramentas matemáticas de produção dessa ciência - e, da incompletude de toda teoria que contenha os axiomas de Peano. Apresentamos também a perspectiva de Wittgenstein em relação ao teorema da incompletude de Gödel. Por fim, entendemos que esse teorema é compreendido pela comunidade matemática como mensageiro da incompletude como uma característica inerente à axiomatização, não como um impedimento para o prosseguimento da atividade com sistemas formais e sim como um resultado revigorante para a Matemática.

PALAVRAS-CHAVE: Teorema da Incompletude de Gödel (GIT). Bourbaki. Wittgenstein. Matemática. Método Axiomático. Metamatemática.

Introduction

How the mathematical community received Gödel's Incompleteness Theorem (GIT)?[4] It is the question one seeks to answer in this article. The possibility established by GIT for Mathematics was certainly a topic debated by mathematicians of the early 20th century, especially by Bourbaki group[5], which aimed to organize all existing mathematics, structuring it.

In the field of Mathematics Education, one can also ask this question: is there (and what would it be) any relationship between GIT knowledge and Mathematics Education? In this regard, Batistela (2014) and Batistela & Bicudo (2018) explain a view of how the knowledge of incompleteness theorem can shed light on the conception of mathematics and consequently influence mathematics teaching.

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[4] "The GIT, or Gödel's Incompleteness Theorems, as theorems VI and XI exposed in the incompleteness theory presented in Gödel's article became known, demonstrates (first theorem) "that formal arithmetic, and by extension most of the mathematical theories interesting, it was incomplete (and, worse, incompletable)." (DA SILVA, 2007, p. 204) [...]; and (according to the theorem, a corollary of the first one) "that the demonstration of the consistency of formal arithmetic was impossible by methods that could be formalized in formal arithmetic itself." (DA SILVA, 2007, p. 204-205). In Gödel's original article, we find the proof of the first theorem and an argument from the proof for the second." Batistela (2017, p. 35).

[5] Nicolas Bourbaki is the collective pseudonym of a group of French mathematicians who aimed to base all mathematics on the theory of sets. The founding members were: Henri Cartan, Claude Chevalley, Jean Delsarte, Jean Dieudonné and André Weil. The group worked for more rigor and simplicity, creating a new terminology and concepts, over time. The group wrote a series of books that began to be published in 1935 and exposed the Modern Advanced Mathematics so that they could serve as a reference for students and researchers. Although the Bourbaki Group is officially known as the Nicolas Bourbaki Collaborators' Association, here in this text we will treat Bourbaki as a person. To learn more about Bourbaki, you can, among other titles and works, read *Scientific American Brasil* – Coleção Gênios da Ciência, 2012, p. 68-98.

The reason that leads us to disclose this subject, the reception and reactions to the GIT in/by the mathematical community, permeates our comprehension that the knowledge of the history of mathematical ideas, especially related to this theorem, is important, both for undergraduates and bachelors in mathematics, given that both often return to universities acting as professors in undergraduate courses. We are aware that it is possible to understand a theorem at different levels. For the pursued effect, we emphasize that the knowledge of the GIT must be at a level that leads to the understanding of the power of the axiomatic method and the structure of its message, which refers to the incompleteness of the theories that contain the axioms of Giuseppe Peano arithmetic (1858-1932)[6] in their formal systems.

The comprehension of Gödelian incompleteness phenomena can certainly raise doubts in students with a rooted understanding that Mathematics is sovereign over all subjects. It can also break with the idea of terminality of Mathematics that was present in the speeches of mathematicians in the early 20th century (and still remains), given the confrontation of the History of Mathematics to present the perspective that problems and new problems maintain the vigor of research in Mathematics.

Although GIT was produced and addressed by/to mathematical science itself, in the position of mathematical educators, the theorem indicates to mathematicians the scope of the production methods of that science and consequently what they consider as Mathematics. The conception of mathematics that resides within the scientific field of mathematics education is shared in the core.

The establishment of GIT and the dissemination of its results occurred concomitantly with the work of mathematicians in the search for a foundation for Mathematics. One of the exponents of this endeavor was David Hilbert (1862-1943) who developed a program that aimed the axiomatization of all Mathematics through Logic, removing the semantics of mathematics discourse, making it axiomatized, with pure manipulation of symbols, through finitary reasoning and with a formal symbolic system. He believed that in this way he would show that mathematics would be free of contradictions.

GIT was published in the article Uber formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, Gödel (1977), in 1931. He shows

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[6] A description of Peano's axioms: i) *Zero* is a number; ii) If *a* is a number, the *successor of a* is a number; iii) *Zero* is not the successor to any number. iv) Two numbers that have equal successors are the same; and v) If a set of numbers *S* contains *zero* and also *the successor to all number of S*, then all number is in *S*.

that the axiomatization method – until then considered to be unlimited and immune to internal contradictions – appears, in Nagel and Newmann words, “partly corrupted by an effect of self-inefficiency” (NAGEL & NEWMANN, 1973, p.19).

Through this demonstration, from the basic arithmetic of natural numbers, Gödel proved the impossibility of demonstrating certain important propositions in arithmetic, and, as a result, confirmed that logical systems containing Peano arithmetic will never establish their consistency internally.

The structure idealized and in a state of construction by Hilbert school – the Formalism which supported the expectation that many branches of mathematics could be free from internal contradictions – had been shaken by the repercussion of the incompleteness theorem. Thus, research on foundations of mathematics, which fed on this hope, soon lost their strength and the mathematicians' efforts diverged. GIT has shown that there are, and there will always be, mathematical truths that are impossible to demonstrate formally. Thus, a logical system such as mathematics cannot be sufficient on its own; it cannot be based on itself.

GIT demonstration proves the presence of undecidable propositions in natural numbers arithmetic (first incompleteness theorem) and, consequently, the impossibility of this theory demonstrates its own consistency (according to incompleteness theorem). The point of impact of Gödel's incompleteness theorem in mathematics was Hilbert's second problem [7], which requested a proof of the consistency of arithmetic. Hilbert's program –the one that aimed to base Mathematics on the basic arithmetic of natural numbers – depended on demonstrating the second problem to complete the program. The GIT announces the impossibility of proving the consistency of the arithmetic in the arithmetic itself. This caused a setback in Mathematics, because at that time, Formalism was the third of the philosophical schools of Mathematics that continued to seek to substantiate Mathematics, as Logicism and Intuitionism schools had already understood the impossibilities of their projects for the foundation of Mathematics. Mathematics, each with its own motive. The logicist school worked to translate all math already made into logical expressions, and the intuitionist school tried to rewrite all the existing mathematics by eliminating non-constructive proofs, that is, the proofs that demonstrated the existence of mathematical objects by reducing to the absurd, of mathematics and Logic.

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[7] Hilbert´s second problem is stated as follows: Demonstrate the consistency of the axioms of arithmetic.

Gödel´s own interpretation to the impact of his incompleteness theorem was launched in 1933, two years after the lanching of GIT, Gödel (1933), in an article which Gödel indicates the existence of ways for Hilbert's program to be reissued, however, it is a fact that the incompleteness theorem impacted Hilbert's program. In the opinion of Da Silva (2003):

> In any case, Hilbert's program was certainly substantially weakened by Gödel's remarkable results. However, it did not die, and Gödel himself contributed to a modified version of it, namely, establishing by appropriate constructive means (finite, predicative, intuitionist, etc.) the relative consistency of formal theories in which parts of classical mathematics can be developed[8]. (DA SILVA, 2003, p. 35).

Referring to the meaning of *incompleteness*, Nagel & Newman explain:

> The axioms of a deductive system are “complete” if every true statement that can be expressed in the system is formally deducible from the axioms. If this is not the case, that is, if not every true statement expressible in the system is deducible, the axioms are “incomplete”. (NAGEL; NEWMAN, 1973, p. 83).

In the demonstration developed by Gödel, ***G***[9], the undecidable proposition, is a truth arithmetical formula obtained by a metamathematical argument and, so, it is not formally deductive in this theory. Therefore, in the hypothesis that the set of axioms of arithmetic is consistent, it follows that it, the set, is incomplete.

Once the set is incomplete, one can think of adding the undecidable, a truth without proof, to the basis of the theory along with the other axioms as a subsequent axiom. However, the new set added to this later axiom would still be insufficient to formally produce all the arithmetic truths, because another true but undecidable arithmetic formula could be constructed in the new expanded system, in the same way that ***G*** was constructed.

The undecidable forces the recognition of a fundamental limitation in the axiomatic method power because it announces that there are mathematical truths that are beyond the truths derived from the Mathematics way of production.

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[8] Original text: “Seja como for, o programa de Hilbert certamente foi substancialmente enfraquecido pelos notáveis resultados de Gödel. Entretanto, não morreu, e o próprio Gödel contribuiu para uma versão modificada dele, a saber, estabelecer por meios construtivos apropriados (finitários, predicativos, intuicionistas, etc.) a consistência relativa de teorias formais nas quais partes da matemática clássica possam ser desenvolvidas.”

[9] It is important to highlight that Gödel built the formula G that says of itself that it is not demonstrable. It is the mirror image *within* the arithmetic calculation of the metamathematical statement: "The formula with the number of Gödel *sub (n, 13, n)* is not demonstrable." (NAGEL, NEWMAN, 1973, p. 80).

Mathematical structures and the Bourbaki group

Bourbaki´s structures are presentend in Bourbaki (1950)[10]; they are structures this group proposes to Mathematical[11]. One understands that this work defines mathematicians´ task and legitimates this profession because it categorizes the objects that the mathematicians should/will deal with.

Bourbaki (1950) is presented in the mathematical community after the flourishing of ideas and conceptions of what should be the mathematics foundations, that is, the philosophical schools of Logicism, Intuitionism and Formalism that sought this foundation, and after the GIT that hits exactly the formalist school main pillar. Bourbaki´s work argues that mathematics deals with a wide range of themes and that, since the 19th century, the number of publications on these subjects has increased. Furthermore, it observes that the work of mathematicians is carried out in stagnant domains within the scope of mathematics itself. Thus, presenting a vision of mathematics as a whole – as a scientific field that covers all topics – is an almost unavoidable task. Aware of this, one takes up the challenge that articulates the points of view until the general presentation.

About the distribution of mathematicians in mathematics, Bourbaki (1950) highlights:

> Many mathematicians take up quarters in a corner of the domain of mathematics, which they do not intend to leave; not only do they ignore almost completely what does not concern their special field, but they are unable to understand the language and the terminology used by colleagues who are working in a corner remote from their own” (BOURBAKI, 1950, p. 221)

Due to the isolation of mathematicians producing mathematics, each one in their own domain, the group raises the issue of whether there is one mathematics or several mathematics as, although the transit of mathematicians through different domains is allowed, this rarely happens. Likewise, it questions whether the exuberant proliferation of mathematical production makes this science a stronger and cohesive organism and in unity with its new developments, or whether there is a trend towards a progressive fragmentation in which disciplines are separated from each other in objectives, methods and different languages.

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[10] This manifesto was written in 1948 by J. Dieudonné, on behalf of the group, and defends the construction of mathematics on structures of different types. Roque affirms that "The metaphor that an ‘architecture’ was being proposed clarifies a lot about the author's desire to build a unified theory that, like a building, rests solidly on its foundations". (ROQUE, 2012, p. 475).

[11] “In this work there is, again, no mention of Gödel, but on this occasion there is a suggestion of difficulties that mathematics will have to overcome”. (MATHIAS, 1992, p. 5).

Following the argument, the Bourbakists observe that the common aspect to all mathematical production is revealed in the procedures employed, which are, the formal systems and the axiomatic method; this latter being the one that has brought the closest unity among the different areas. The group understands that after the failure of Logicism, Intuitionism and Formalism projects, which looked for different systems that characterized mathematics as a science marked by a specific definitive method, there was a tendency to look at it as "a set of disciplines based on particular concepts, exactly specified ... connected by a thousand communication roads, allowing the methods of any of these disciplines to fertilize the others.” (BOURBAKI, 1950, p. 223).

With this argument as a basis, the group states that understands all mathematical theory is a concatenation of propositions, each one derived from the precedents, in accordance with the rules of a logical system, conveniently adapted to the particular goals of the mathematician. It explains that deductive reasoning is not a unifying principle for mathematics, even though, superficially, it is thus understood. It argues that the fact that the different branches use the same method, through chains of syllogisms, this cannot be the unifying axis for this science, as it is the external form that the mathematician gives to his/her thought, the vehicle that makes it accessible to the others. However, the axiomatic method provides the intelligibility of Mathematics, which starts from *a priori* belief in the conviction that, in the statements, this method:

> (…) will try, in the demonstrations of a theory, to separate out the principal mainsprings of its arguments; then, taking each of these separately and formulating it in abstract form, it will develop the consequences which follow from it alone. Returning after that to the theory under consideration, it will recombine the component elements, which had previously been separated out, and it will inquire how these different components influence one another. There is indeed nothing new in this classical going to-and-fro between analysis and synthesis; the originality of the method lies entirely in the way in which it is applied.” (BOURBAKI, 1950, p. 223-224)

According to what was exposed, Bourbaki (1950) presents the mathematical structures – algebraic, of order and topological – as a proposal to standardize mathematics through the language of set theory.

It´s important to recall that the main characteristic of the axiomatic method is a significant economy of thought and considering that the structures proposed by Bourbaki are established in order to offer tools for the mathematician, as they allow the mathematician to use general theorems that belong to a structure, recognizing, among the studied elements, relations that satisfy the axioms of another structure.

This reveals the way in which mathematics has been produced mainly after the failure of the projects that aimed at its foundation and after the GIT. Note that, for this author, the work of the mathematician, before this standardized way proposed by himself was carried out as the mathematician "was obliged to forge for himself the means of attack on his problem; their power depended on his personal talents and they were often loaded down with restrictive hypothesis, resulting from the peculiarities of the problem that was being studied." (BOURBAKI, 1950, p. 227). However, Bourbaki reflects:

> The mathematician does not work like a machine, nor as the workingman on a moving belt; we can not over-emphasize the fundamental role played in his research by a special intuition, which is not the popular sense-intuition, but rather a kind of direct divination (ahead of all reasoning) of the normal behavior, which he seems to have the right to expect of mathematical beings, with whom a long acquaintance has made him as familiar as with the beings of the real world." (BOURBAKI, 1950, p. 227).

In this excerpted it would be the main argument of the group's proposal, which, in our understanding, would facilitate the work of mathematicians in the sense of showing them that there are structures that bring their productions together, which until then had been developed in isolation.

Explaining more precisely what a structure is about, Bourbaki (1950) states that this idea can be applied to sets of elements whose nature is not specified, because in the definition of a structure, one or more relationships between these elements are taken as given. Then, a certain relationship or relationships are postulated to satisfy the axioms of the structure in question. In order to configure the axiomatic system of a given structure, the axioms of the structure are elevated to the consequences of logical deduction, disregarding any hypothesis about the elements of the set in question, as well as their own nature. Then, he explains that "each structure carries with it its own language, freighted with special intuitive references derived from the theories from which the axiomatic analysis described above has derived the structure" (BOURBAKI, 1950, p. 227). From this, he justifies that in the proposal of the structures, the mathematicians end up having at their disposal powerful tools provided by the great types of structures;

> What all this amounts to is that mathematics has less than ever been reduced to a purely mechanical game of isolated formulas; more than ever does intuition dominate in the genesis of discoveries. But henceforth, it possesses the powerful tools furnished by the theory of

> the great types of structures; in a single view, it sweeps over immense domains, now unified by the axiomatic method, but which were formerly in a completely chaotic state" (BOURBAKI, 1950, p. 228)

Referring to the mathematical work after GIT, Morris Kline (1980), compares the mathematician to a land cleaner who, when cleaning, realizes the presence of wild animals hidden in the woods around him and even cleaning a larger area, he knows he just chased these animals away. Animals understood here by us as being the undecidable problems, which Gödel's has proven the existence and can one day be found.

In view of this, we understand that the message that stands out for the work of mathematicians is that, to the certainty of the presence of wild animals, is added the uncertainty regarding the bifurcations that ask for choices, however, when they are made, they show paths for the continuation of mathematics production activity.

Bourbaki´s reception to GIT

The acceptance of the result of incompleteness in Mathematics, in our understanding, is revealed in the attitude of the Bourbaki group of having realized the possibility of meeting with undecidable problems, while they claim that they would continue with the ideal of a complete formalization of mathematics, and also in the reflection that they developed and showed to be knowledgeable about the options facing the encounter with an undemonstrable proposition.

The mention of Gödel's name in this work mentioned above appears on page E.L 12 in the third paragraph, in a comment about the impossibility of proving the arithmetic consistency:

> To escape this dilemma, the consistency of a formalized language would have to be "proved" by arguments which could be formalized in a language less rich and consequently more worthy of confidence; but a famous theorem of metamathematics, due to Godel, asserts that this is impossible for a language of the type we shall describe, which is rich enough in axioms to allow the formulation of the results of classical arithmetic (BOURBAKI, 1968, p. 12)

Continuing the presentation of the ideas that guide his works, Bourbaki (1968) argues, in relation to the theory of sets, that, from the relative proofs of consistency, which logically connect the various mathematical theories to the theory of sets, it follows that any contradiction found in some theory must give rise to a contradiction in set theory. However, for this reason, one cannot deduce the consistency of theory of sets. Still in this topic:

> Nevertheless, during the half-century since the axioms of this theory were first precisely formulated, these axioms have been applied to draw conclusions in the most diverse branches of mathematics without leading to a contradiction, so that we have grounds for hope that no contradiction will ever arise (BOURBAKI, 1968, p. 13)

Bourbaki comments that if the contradiction comes from other ways, this would be inherent to the fundamental principles of theory of sets, a fact that requires changes, similarly to what happened when the paradoxes in this theory emerged and this was revised. Therefore, this formalized language was adopted equivalent to the description in the work *Elements of Mathematics: Theory of Sets* Bourbaki (1968). Bourbaki uses the proposal of structuring and formalization of mathematics and, continuing, presents that they will face the future of this science with serenity, because they understand that mathematics, in more than two thousand and five hundred years of existence, has been correcting its errors and enriching itself. Based on the experiences with overcoming mathematics itself, the Bourbaki group believes that mathematics is destined to remain alive, even if a contradiction arises at some point.

With the group´s explanation, we understand that mathematics develops and solves some problems and perceives the existence of others. Thus, there is an implicit consideration for the GIT, taken by them as an express data that affirms something more than just the observed experience, but that will be circumvented or overcome at some point.

We understand that, for Bourbaki, the threat brought by the GIT was so far from the basis of the theory that it could be ignored. In this way, the bourbakists, aware of the proof of incompleteness, continued to be guided by the ideal of perfect rigor in their work and of the possibility of accomplishing the complete formalization of mathematics. Changeux and Connes (1996), regarding the meaning of GIT, explain:

> The theorem only states that, with a finite number of axioms, we cannot have an answer for everything. However, if a question is not decidable, on condition that you have demonstrated it, we can give it an answer and continue to reason. This means that each new undecidable question leads to a bifurcation, from the moment we choose a positive or negative answer. The world in which we move involves several possible bifurcations. That is all of its meaning. Once an answer is given to the question, we can continue and ask ourselves new questions. Ancient issues that were not decidable then become... each undecidable question creates a bifurcation and imposes a choice. (CHANGEUX; CONNES, 1996, p. 174).

One must recall that Gödel, when presenting the **G** formula – truth and undemonstrable – announced the existence of a non-empty set of undecidable propositions. However, no sample was found in specie in any theory. The presence of an undecidable problem, within the scope of a theory, indicates the non-contradiction of that theory. In GIT demonstration, it follows that the non-contradiction of Peano arithmetic implies the undecidability of a ***G*** proposition and, conversely, the undecidability of a ~***G*** proposition guarantees the non-contradiction of Peano arithmetic, given that, from a contradictory theory it is possible to deduce every expressible proposition in the language of that theory. In Gödel's proof, undecidable ***G*** means "that in arithmetic and, more generally, in any axiomatized theory that is not contradictory and rich enough to contain the basic arithmetic of naturals, the non-contradiction of the theory itself is not demonstrable in the language of the theory" (GUERRERIO, 2012, p. 50).

Still on how this result was received in mathematics, it is necessary to explain that there are controversies about Bourbaki having understood and accepted the result of the GIT with attention and understanding equal to the importance of this theorem. Mathias (1992), in relation to the reception and reference of Bourbaki to the GIT, indicates the group's posture as the one that ignored this result. This author presents a study of the main Bourbaki´s works from the 1930s and 1940s and, going through texts by Henri Cartan, Jean Dieudonné and André Weil, interprets that Bourbaki demonstrates an absence of understanding of the different meanings of *truth* and *demonstrable* treated by Gödel. For Mathias, this reveals a skeptical awareness of Gödel's results, assuming that the reader knows the result. Mathias understands that, in avoiding mentioning the name or considering the GIT, Bourbaki did not pay due attention to the result taking into account the vigor of its message.

Related to the effect that Gödel´s theorem has in Bourbaki, Mathias shows:

> One might almost say that they ignored him, except that the tone of certain of their works suggests a conflict between an uneasy awareness that something has happened and a desire to pretend that it has not. It is as though they had discovered that they were on an island with a dragon and in response chose to believe that if the dragon were given no name it would not exist (MATHIAS, 1992, p. 6)

Mathias (1992) understands that Bourbaki's attitude does not consider Gödel's important contribution to fundamental questions and he questions the reason. In the structure of his exhibition, he comments and presents curiosities, but in the end, ponders that he has no sociological or psychological explanation about Bourbaki's resistance to the result of the incompleteness established by Gödel. However, he launches a hypothesis: “bourbakists may have been seduced by Hilbert and by his commitment to his program, and this, in principle, could cause great difficulty in accepting Gödel's work” (MATHIAS, 1992, p. 6). Mathias also notes that Hilbert had recovered from the shock more quickly than his possible younger French disciples had. And, he concludes by stating that further explanations about Bourbaki's behavior in consideration of the GIT are necessary, because what is evident, for him, is that the Bourbakists were not ready to face the consequence of the GIT, that is, the possibility that there may not be complete foundations for mathematics, that is, there is no way to circumscribe mathematics.

Nonetheless Mathias (1992) reveals his consideration to Bourbaki´s work and his comprehension that they ignored one of the most invigorating ideas of mathematics. Amongst them, it is the GIT:

> [...] I do not dispute the positive worth of their books nor the magnitude of their achievement; but I suggest that their attitude to logic and to set theory, which has been passed on to younger generations of mathematicians, is harmful because it excludes awareness of perceptions of the nature of mathematics that are invigorating; and I almost venture to suggest that if, as some say, Bourbaki is now dead, he was killed by the sterility of his own attitudes. (MATHIAS, 1992, p. 8)

To what we have been discussing above about Bourbaki's consideration of the GIT, Mashaal (2012) states that Bourbaki pretended to be an ostrich, referring to the group's attitude on the axiomatization of the theory of sets, to the research on the foundations of mathematics and to Gödel's demonstration that, whichever system of axioms is chosen, it is impossible to demonstrate the non-contradiction of mathematics, which results from these axioms by using the axioms themselves. Thus Mashaal (2012) says:

> Faced with the “crisis of fundamentals” that brought down mathematics in the first half of the 20th century, Bourbaki chose to pretend to be an ostrich and consider the metamathematical problems that plagued logicians to be uninteresting. It is difficult to understand, however, that the logical coherence of an axiomatic theory may be an issue of disinterest to a mathematician who, like Bourbaki, seems to attribute so much importance to the axiomatic approach. This somewhat schizophrenic attitude by Bourbaki - shared, let us say in passing by most mathematicians who do not work directly with the foundations of their discipline - appears concretely translated in the 'book' of theory of sets in *Elements of Mathematics*. This book was severely criticized, especially by logicians, because of its excessively narrow focus and because it obscured the issue of fundamentals. (MASHAAL, 2012, p. 96).

It is controversial the discussion about whether Bourbaki was dead. We believe that his works, even though he did not accept all the consequences of the

GIT, reflect the way in which mathematics has been done after this theorem and we cannot fail to recall that this is how mathematics appears in the books of specific disciplines in mathematics undergraduate courses.

Wittgenstein´s reaction to the GIT

As we became aware of Gödel's sense of humor, we could understand his attitude regarding many comments made by Ludwig Wittgenstein (1889 - 1951) of objection to the Gödel's incompleteness results. These comments, according to Goldstein (2008, p. 100), "were the type that causes resentment, even though there was no [resentment][12] before".

Now, a glimpse into the heterogeneity between these two geniuses of radically unequal styles can give us a perspective of understanding about Gödel's prudence: Gödel had already attended Vienna Circle invited by Professor Hans Hahn (1879 - 1934) and behaved with "his hermetically sealed genius, allowing almost nothing of his elevated mind to manifest", (GOLDSTEIN, 2008, p. 97), he "used to watch without saying a word, from when he joined the Circle to when he had the rigorous proof which had spoken on his name about the incompleteness of mathematics" (p. 97). Gödel, in Goldstein's perspective (2008), saw the emergence of the genius Wittgenstein and witnessed the *bewitchment* he caused in the members of the Circle.

It is intriguing to imagine the encounters of the Vienna Circle with the differences in views and styles of genius between Gödel and Wittgenstein. In the life-world we share, as researchers in the Mathematics Education community and in our universities, we certainly have experiences that allow us to imagine which human emotions would have incited the silent dissident that confronted the philosopher's divine inspiration with a greater authority: mathematics.

However, our imagination is nothing more than conjecture "given the opacity of Gödel's interior life" (GOLDSTEIN, 2008, p. 97). Regarding his theorems of incompleteness, Gödel spontaneously exposed that Wittgenstein's work had no influence on them, moreover, Gödel let arise claims that Wittgenstein would not have understood or would have pretended not to understand his theorems.

Goldstein (2008) brings the question of Wittgenstein's influence on the GIT in these terms:

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[12] [...]Our addendum. Other aspects of the common experience period between Gödel and Wittgenstein in the Vienna Circle can be accessed in Goldstein (2008) and Levin (2009).

> Obviously, the influence, in a positive sense, is quite different from the kind of more obscure incentive I am speculating about. [...] the influence of the charismatic philosopher on the members of the Circle may have irritated him, amused him (more doubtfully) or even helped him to be inspired in the direction of his test: it is impossible to know. (GOLDSTEIN, 2008, p. 100)

Mere speculation, of course, but it seems interesting because this brings a possible human perspective to the situation, an underlying motivation, often not considered, but highly likely also in our academic environment. We frequently go through ebullition with reflections on the incentive that what we oppose can provoke in scientific productions.

The question of the scope of the incompleteness results was the point of greatest divergence between the two in relation to GIT. Wittgenstein argued contrary to Gödel regarding theorems. The question between them, after GIT, was about the scope of the result and Wittgenstein never admitted that Gödel had reached a result, through mathematics, with metamathematical implications.

The incompleteness results went against the Wittgensteinian concept of language, knowledge and philosophy. Wittgenstein revealed his comprehension that:

> The mathematics cannot be incomplete, as neither a sense can be incomplete. What I can understand I must fully understand. This is in accordance with the fact that my language works the way it is, and that logical analysis does not need to add anything to the present sense in my propositions to reach the complete clarity. (GOLDSTEIN, 2008, p. 160)

Wittgenstein stated that Gödel's incompleteness theorems were logical tricks, thus depriving them of the metamathematical importance attributed by Gödel – and in/by the mathematical community – to the result, repudiating it in an extremely unpleasant manner. Goldstein affirms that the adjective *unpleasant* that characterizes the attitude of this philosopher in reception of the incompleteness theorem was a general opinion among mathematicians. And he dares to speak…it was probably also Gödel's (GOLDSTEIN, 2008).

In Goldstein's (2008) words about Wittgenstein's inflexibility in denying the possibility of a proof like the incompleteness theorem: "no calculation can solve a philosophical problem. A calculation cannot give information about the foundations of mathematics" (GOLDSTEIN, 2008, p. 160). Wittgenstein stated that he would no longer speak about GIT, but in the book *Remarks on the Foundations of Mathematics*, he tried to show that the meaning of the incompleteness theorem is in conflict with his philosophy.

Wittgenstein's views on mathematics – mainly that the meaning of an arithmetic generalization is its proof are controversial. Wittgenstein's comments on the incompleteness theorems, which Gödel himself said were “an erroneous interpretation, totally trivial and uninteresting” (GOLDESTEIN, 2008, p. 100). According to Silva (2018, p. 99), the logician and mathematician Georg Kreisel, a former student of Wittgenstein, who stated in a review of *Remarks on the Foundations of Mathematics* “Wittgenstein's views on mathematical logic are not very valuable , because he knew very little about the subject and this little was restricted to the luggage of the Frege – Russell lineage ”(MONK, 1995, p. 441 *apud* Silva (2018)).

Although Wittgenstein's claims are unfounded, as Gödel himself and some Wittgenstein scholars are aware and expressed about it, the confrontation between Wittgenstein and Gödel regarding the validity and scope of the incompleteness result has been extensively explored and there is enough bibliography. On a personal level, perhaps Wittgenstein has been the only opponent who expressed himself at the level of denial of acceptance of the theorem.

It is important to highlight that the way Wittgenstein reacted to the result of incompleteness was different from how Hilbert understood and received the GIT, although it was indigestible to his program and his philosophical perspective. While Wittgenstein did not even admit the existence of the result, he did not even recognize the validity of the result, a properly demonstrated mathematical theorem, Hilbert felt dissatisfied with the result that frustrated the expectations of his project which had proof of the consistency of Peano's axioms, or that is, with proof that the objects defined by such axioms exist, but the GIT has established that such proof cannot be performed in arithmetic.

Understanding the reception of GIT in Mathematics

The receptivity and coexistence with the GIT in Mathematics is shown in the attitude assumed by the Bourbaki group when facing the fact that the set of undecidables is non-empty – in theories that contain arithmetic – and deciding that, despite this, the activity of mathematical production would continue to be carried out considering the characteristic of the impossibility of simultaneously obtaining the consistency and completeness of simple theories that deal with the arithmetic of natural numbers.

In the midst of the introductory question about the possibilities of continuity of mathematics in the face of the GIT result, Bourbaki in *Eléments*[13] *de Mathématique: Théorie des Ensembles,* reveals that, even having in mind the incompleteness results, he would make use of the axiomatic method in the development of his enterprise of structuring the mathematics, and he will always bear in mind the possibility of a complete formalization of theories and with a perfect rigor. Adiction, it would not allow that this – the contradiction found – to remain without disabling the theory in which it occurred useless. This work, Bourbaki (1950), to which we refer, is the first of several books by Bourbaki and this statement appears in the introduction of seven pages reserved for the presentation of the method that the group proposed to follow and the beliefs and concepts that moved them in the proposal in diffusion.

Wittgenstein´s attitude, who strongly repelled the validity of the incompleteness theorem, stating that a mathematical result could not have a philosophical scope in relation to the foundations of mathematics, disregards the fact that the GIT is a theorem of arithmetic demonstrated in the sphere of metamathematics and that allows metamathematical discussions and relations between mathematics and Logic, thus reaching the foundations of mathematics.

By exposing our understanding of how the mathematical community understands mathematics after the GIT, we are exposing how mathematicians embraced Gödel's incompleteness theorems, which is a mathematical result that demonstrates that the objects defined by Peano's axioms do not exist. Note that the consistency of Mathematics is intuitively logical, this is a consensus among mathematicians, according to Abrahão (2011). From the logical mathematical point of view, Gödel's incompleteness theorems are valid mathematical discourses. The acceptance of the demonstration of incompleteness means dealing with two things: 1) that there are arithmetical truths that cannot be reached by mathematical logic; and, 2) that the consistency of the arithmetic cannot be demonstrated. In other words, it means accepting that something intuitively obvious cannot be formally attested, without ultimately resorting to a non-mathematical argument, the intuition.

We shall understand: Gödel demonstrates that arithmetic has a property of not being able to demonstrate that all the objects it deals with exist, that is, GIT explains

[13] "It was a textbook to teach mathematical analysis on a new basis. The title *Eléments* already indicated the desire to codify the styles of mathematics according to the pattern defended by the group, but gradually the enterprise was extended to understand all branches of mathematics, instead of the diversification of methods and objects that had prevailed in mathematics until that moment." (ROQUE, 2012, p. 475).

that there are truths in arithmetic that escape mathematical methods of proving the existence of mathematical objects. In this context, questions arise about, mainly, how Gödel demonstrated that such a property cannot be demonstrated. This is not the case in this article. We understand that this property can be disturbing, raising doubts about the demonstration carried out and how to understand it within the scope of the results that inherit that property.

Mathematicians assume that the incompleteness theorem carries a vigorous message from mathematics and this occurs when they understand that the consistency of a system is to know how to deal with paradoxes, when and if they arise in the system. The GIT makes it clear that theories dealing with natural numbers are unable to prove all the truths that can be established in the theory, which we understand to mean that Gödel renames *undecidable* and *contradiction*. The GIT establishes that incompleteness is inherent to axiomatization, in view of this the human process of recognizing the evidence that contradictions were inherent to the axiomatic system, while the philosophical schools, with their projects of rewriting mathematics in their own way, sought to root out the contradictions.

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