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# Problem solving involving algebraic thinking: an experiment in the 9th grade of elementary education

## Resolução de problemas que envolvem o pensamento algébrico: um experimento no 9º ano do ensino fundamental[1]

Giovani Rosa Delazeri [2]

Claudia Lisete Oliveira Groenwald[3]

## ABSTRACT

This article presents the results of a master dissertation aimed at investigating whether 9th-grade students of a state elementary school of the city of Porto Alegre, Rio Grande do Sul, have developed the competence to solve problems that involve algebraic thinking in the contents of first degree equations and systems of equations of the first degree. The experiment was based on adaptive tests applied to identify the level of algebraic thinking of the respondents. The results indicate that the topic in which the investigated students were more successful was mathematical language, where they should check the alternative that contained the numerical expression corresponding to the proposed problem, while the topic in which they presented greater difficulty involved systems of equations, where students were able to assemble the system but failed to solve it.

**KEYWORDS:** Problem solving, Algebraic Thinking, Elementary School.

## RESUMO

Este artigo apresenta os resultados de uma dissertação de mestrado cujo objetivo foi investigar se os alunos de uma turma do 9º ano do ensino fundamental de uma escola estadual do município de Porto Alegre, no estado do Rio Grande do Sul, possuem desenvolvida a competência de resolução de problemas que envolvem o pensamento algébrico nos conteúdos de equações do 1º grau e sistemas de equações do 1º grau. Desenvolveu-se um experimento com esses estudantes com a aplicação de testes adaptativos, buscando-se identificar o nível do pensamento algébrico dos respondentes. Os resultados apontam que o tópico em que os alunos investigados demonstraram maior facilidade foi o de linguagem matemática, onde eles deveriam assinalar a alternativa que continha a expressão numérica correspondente ao problema proposto, e o tópico em que apresentaram maior dificuldade foi o que envolvia sistemas de equações, onde os estudantes conseguiram montar o sistema mas erraram em sua resolução.

**PALAVRAS-CHAVE:** Resolução de Problemas, Pensamento Algébrico, Ensino Fundamental.

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[2] State Network of the state of Rio Grande do Sul. giovani_matematica@yahoo.com.br

[3] Universidade Luterana do Brasil. claudiag@ulbra.br

## Introduction

This paper presents the results of a master's dissertation aimed at investigating whether 9th-grade students of elementary school have developed the competence of solving problems involving algebraic thinking with the contents of 1st degree equations and systems of 1st degree equations.

For this, the Integrated System of Teaching and Learning (SIENA) was used, with the application of adaptive tests on mathematical problems that deal with subjects related to elementary algebra and the contents of first degree equations and systems of first degree equations. SIENA is a computer system developed jointly by the groups of the Curriculum Studies in Mathematics Education (GECEM) of ULBRA and Educational Technologies of the University of La Laguna, Tenerife, Spain.

This study highlights the importance of algebraic thinking for elementary school students to solve problems. For Souza (2013, p. 23), algebra is currently used in a mechanic way, without the students fully understanding the reason for the algebraic calculus.

According to Dalton and Buriasco (2009, p. 452), the low achievement in mathematics, mainly with algebraic contents, seems to be related to the way algebra is worked in schools. Ameron (2002) states that in schools traditional algebra is often presented to students as a rigid, abstract system, with little connection to the real world.

According to Groenwald (1999), the scarce ability of most young people and adults to solve problems is visible, and the lack of training in skills and patterns of general and useful strategies to solve problems is emphasized.

Thus, this research seeks to deal with both themes in an integrated way, exposing what authors understand by algebraic thinking and problem solving.

## Algebraic thinking

For Godino and Font (2003, p. 774), the teacher must recognise the importance that algebra and algebraic thinking have for mathematics, stating that algebraic reasoning implies representing, generalising and formalising patterns and regularities in any aspect of mathematics. And as this reasoning develops, it is evolving in the use of language and its symbolism, necessary to support and communicate algebraic thinking, especially in equations, variables, and functions.

Ribeiro (2015, p. 11) says that: "algebra should be explored from the earliest years of teaching, as it forms part of a set of processes and thoughts that originate from experiments with numbers, patterns, geometric entities and data analysis".

For Fiorentini, Miguel and Miorin (1993, p. 88): "algebraic thinking can be developed gradually even before the existence of a symbolic algebraic language". Those authors also deal with both the classical conception, formatted from a universal arithmetic, and the modern conception of algebra, based on arbitrary symbolic principles.

Kieran (1992) classifies algebra as procedural and structural, in which the former does not deal with algebraic expressions properly, but rather with arithmetic operations, and the second concerns to the use of algebraic expressions containing a numerical part and a literal part, resulting in algebraic expressions. As exemplified in frame 1, adapted from Ponte, Branco and Matos (2009).

Frame 1 - Examples of classification of algebra into procedural and structural

| **Procedural** | **Structural** |
|---|---|
| • Immediate replacement of variables by numbers<br>• Performing arithmetic operations | • Using conventions that are proper of the structure of algebraic expressions |

Source: Adapted from Ponte, Branco and Matos (2009, p. 78)

According to Becher (2009), both procedural and structural perspectives are worked separately in most textbooks. Frame 2 presents the framework of competences and skills mapped between procedural and structural.

Frame 2 - Algebraic competencies and abilities mapped and divided between procedural and structural

| **Procedural algebra** | |
|---|---|
| **Algebraic competence** | **Algebraic skill** |
| Understanding algebraic representations | Reading algebraic representations |
| | Representing algebraic relations |
| | Understanding and representing algebraically |
| | Understanding and expressing algebraically |
| Operating algebraically | Using formulas |
| | Numerical value |
| | Resolving 1st degree equations |
| | Resolving 2nd degree equations |
| **Structural Algebra** | |

| Algebraic competence | Algebraic skill |
|---|---|
| Operating algebraically | Properties and operations with N |
| | Operating algebraically |
| | Algebraic properties and operations |
| | Resolving systems and inequalities |
| | Properties and operations with R |
| | Understanding and using algebraic properties |
| Recognising and representing patterns | Recognising patterns |
| | Creating representations |
| | Generalising and deducing formulas |
| Solving problems | Algebraic Problems |

Source: Becher and Groenwald (2010)

This classification may owe to what Godino and Font (2003) call a traditional view of school algebra, called "generalised arithmetic", which is considered only as a manipulation of letters that represent indeterminate numbers. So, while arithmetic would use numbers and numerical expressions, in which numbers combine with symbols, algebra would use numbers as variables, unknowns, represented by letters or expressions, although operations, as the basic rules used by both, may be the same.

According to Radford (2011), the mathematical knowledge developed around problem-solving activities may bring some insight into how to introduce and structure algebra in school, leading us to re-thinking, in a new perspective, the role of problems in the teaching of algebra.

For Falcão (1997), algebra is a set of procedures that serve to represent and solve certain math problems that arithmetic only would not solve. However, the passage from one to the other brings the students many discomforts and problems, and can even interfere in their learning, so the author states that "algebra is more than the generalisation of arithmetic".

Ponte (2006) emphasises the difficulties students have in the transition from arithmetic to algebra: giving meaning to an algebraic expression; not seeing the letter as representing a number; assigning concrete meaning to the letters; thinking of a variable with the meaning of any number; understanding the changes of meaning in arithmetic and algebra of the symbols "+" and "=" and, in particular, distinguishing arithmetic addition (3+5) from the algebraic addition (x +3).

Algebra has a dual function, according to Falcão (2003, p. 31): "representing phenomena and relations and helping solve problems", according to frame 3.

Frame 3 - Basic elements of characterization of the conceptual field of algebra

| **Activities in algebra** | |
|---|---|
| **Representational tool** | **Problem-solving tool** |
| Modelling: capture and description of the phenomena of the real.<br>Function: Symbolic explanation of elementary relations.<br>Generalisation: Passing from specific descriptions linked to a context to general laws. | Algorithms: syntactic rules, priorities of operations, principle of equivalence between equations. |
| **Basic elements of the algebraic conceptual field** | |
| Numbers, measures, unknowns and variables, rules of symbol assignment, range of notions for the equal sign, transit between forms of language. | Operators, syntax, priority of operations, principle of equivalence, knowledge-in-action linked to extracurricular experiences, instrumental arithmetic facts (for example: neutral element of addition). |

Source: Adapted from Falcão, 2003, p. 31

Lins and Gimenez (1997) affirm that the important thing is to understand how algebra and arithmetic are linked, what they have in common, since this would be fundamental to rethink arithmetic and algebraic education in a unique way. So:

> The scope of arithmetic education has hitherto been insufficient, whereas the objectives of algebraic education have been insufficient. (...) For both arithmetic and algebra, the most important change of perspective refers to thinking in terms of meanings being produced within activities, rather than in terms of techniques or contents, as it has been thought up to the moment (LINS; GIMEMEZ, 1997, p. 161).

The idea of a correlation between arithmetic and algebra is reinforced by NCTM (2000, p. 39): "Much of the symbolic and structural emphasis in algebra can be built upon students' extensive numerical experience", although the idea that algebra is not restricted to manipulations of symbols is sustained, since it is necessary to understand them by understanding their concepts, structures and principles.

Going beyond the traditional view that algebraic activity would be "calculating with letters", Lins and Gimenez (1997) investigated the characterisation of the algebraic activity and its peculiar cognitive processes. "Algebra consists of a set of statements for which it is possible to produce meanings in terms of numbers and arithmetic operations, possibly involving equalities and inequalities" (LINS; GIMENEZ, 1997, p. 150).

The algebra of elementary, middle and high school involves understanding the meaning of letters, symbols, and operations. The differences that we will have in the variables will come according to the use that is made, as well as the moment in which they are used, since the concept of variable becomes very vague.

According to the PCN (BRASIL, 1998), during the final years of elementary school (middle school) the algebraic activities should be expanded, developing the skills of generalising, finding algebraic patterns, establishing relationships between two quantities, modelling, solving arithmetically difficult problems, representing them by equations, in which the variables and unknowns are differentiated.

Frame 4 presents a chart proposed by Becher (2009) with algebraic skills and competences developed in middle school, which considered the skills and competences proposed by NCTM[4], ENEM[5], PCN[6] and PISA[7].

Frame 4 - Algebraic skills and competences developed in middle school

| Competences | | Skills |
|---|---|---|
| Understanding algebraic representations | Basic | Recognising representations |
| | | Reading representations |
| | Plain | Expressing ideas and relationships using algebraic representations |
| | | Comparing and relating algebraic representation with different forms of representation |
| | | Understanding the meaning of solutions |
| Operating algebraically | | Determining the numeric value |
| | | Using formulas |
| | | Solving equations |
| | | Performing algebraic operations |
| | | Using algebraic properties |
| | | Resolving systems and inequalities |
| | | Explaining mathematical facts and procedures using algebra |
| Recognising patterns and generalising | Basic | Recognising patterns using numerical methods |
| | | Using tables to represent variations |
| | | Verifying numerical properties |
| | Plain | Recognising patterns of variation |
| | | Expressing relations using functions and expressions |
| | | Verifying algebraic properties |
| Solving problems | | Using algebraic representations in problem solving |
| | | Using algebraic methods and techniques to solve problems |
| | | Elaborating algebraic justifications for problem solving |
| | | Making use of different forms of representation and analysis to solve algebraic problems. |

Source: Becher (2009)

Through this chart we can deduced that for the development of problem-solving skills, for example, the use of algebraic representations, methods and techniques, and different forms

[4] NCTM - National Council of Teachers of Mathematics
[5] ENEM – Exame Nacional do Ensino Médio (High School National Exam)
[6] PCN – Parâmetros Curriculares Nacionais (National Curriculum Parameters)
[7] PISA – Programme for International Student Assessment

of representation and analysis are needed. Algebraic language allows for expressing mathematical ideas more specifically and with greater mathematical rigor.

According to NCTM (2000, p. 97): "when students generalize from observations on numbers and operations, they form the basis of algebraic thinking.”

The Brazilian National Common Curricular Base - BNCC (BRASIL, 2016) - states that some dimensions of the work with algebra, when present since the initial years of elementary education, help in the teaching learning process, founding other dimensions of the algebraic thinking such as problem solving of algebraic structures. The BNCC guidelines state that:

> students work with problems that allow them to give meaning to mathematical language and ideas. When required to solve very diverse problem situations, the student will be able to recognise different functions of algebra (when resolving difficult problems from the arithmetic point of view, when modelling, generalising and demonstrating properties and formulas, establishing relations between magnitudes) (BRASIL, 2016, p. 84).

Blanton & Kaput (2005) consider that algebraic thinking would be like a process in which students generalise mathematical ideas from a set of particular examples, establish this generalisation through the discourse of argumentation, and express it gradually symbolically according to their age.

Continuing in the attempt to establish the domain of the algebraic thinking, which is linked to the development of mathematical thinking, we see Ponte (2009, p. 10): "algebraic thinking includes three aspects: representing, reasoning and solving problems", thus including the domain of contents that should lead to the development of mathematical thinking.

Representing, in this sense, refers to the student's ability to read, understand and operate with symbols, translating these representations reproduced symbolically into other forms, showing meaning and interpretation of the symbol in various contexts. Reasoning is about relating mathematical properties, generalising rules and producing deductions. Finally, solving problems, according to Ponte, means modelling situations, using expressions, equations and systems in the interpretation and resolutions of mathematical problems and other domains.

Ponte, Branco and Matos (2009) represent those three aspects in Frame 5.

Frame 5 - Fundamental aspects of algebraic thinking

| | |
|---|---|
| Representing | - Reading, understanding, writing and operating with symbols using the usual algebraic conventions.<br>- Translating information represented symbolically into other forms of representation (by objects, verbal, numerical, tables, graphs) and vice versa. |

| | |
|---|---|
| Reasoning | - Relating (particularly, analysing properties).<br>- Generalising and acting on those generalisations revealing understanding of the rules.<br>- Deducing. |
| Solving problems and modelling | - Using algebraic expressions, equations, inequalities, systems (of equations and inequalities), functions and graphs in the interpretation and resolution of mathematical problems and other domains (modelling). |

Source: Adapted from Ponte, Branco and Matos (2009, p. 11)

Thus, algebraic thinking is not something simple, but it is composed of different forms of thinking and understanding the symbols and situations presented, being in line with problem solving.

The following chapter presents the problem solving with a methodology favourable to the development of algebraic thinking.

## The problem-solving methodology

In the 1940s, Polya (1995) defined the notion of problem as follows: “having a problem means consciously pursuing an appropriate action to achieve a goal that was imagined, but not immediately achieved", proposing the following steps for resolution: understanding the problem, establishing a plan, executing it and carrying out the retrospective.

First, for Polya (1995, p. 4), understanding the problem is the starting point for resolution, since it states that the problem statement must be well understood and the student must be in a position to identify the main parts, the unknown and the data, asking questions such as: What is the unknown? What are the data presented? Is it possible to achieve the required conditions and are they sufficient to determine the unknown?

In the second step, establishing a plan, students, after understanding what the problem is about, devise strategies that allow this solution, finding connections between the data and the unknown, making a relation with another similar problem so that they can analyse and compare their strategy. It is possible, at this step, to introduce other elements for a better analysis and insight of the problem and to introduce questions such as: Can you see any other statement for the problem using the same data? Can you solve some of the problem?

The third step, executing the plan, is usually considered easier than the previous one, but it depends on that for its success, since the execution requires prior knowledge of various contents. But the development of a wrong strategy will lead to failure, leading to a return to the previous step, and to the need to devise new strategies.

And the fourth and most important step, according to Polya, since it is the closure of the problem, is an analysis of the stages developed until the solution of the problem, trying to identify possible failures, verifying the results and the arguments used to obtain of the solution, verifying both the core of the problem and whether the result achieved satisfies this step.

Kaiber and Groenwald (2008) present, in frame 6, the four steps to be followed, based on Polya.

Frame 6 - Steps to follow for solving problems

| Action steps | Features | Facilitating questions |
|---|---|---|
| Understanding the problem | Steps of reading the statement of the problem to identify data, unknowns and determine what is requested, what elements you have and what elements are missing, what similarities and novelties there are in relation to any other situation already experienced | What is the unknown?<br>What are the data presented?<br>What is the condition? Is the condition sufficient to determine the unknown? Is this sufficient? Redundant? Contradictory? |
| Elaboration of an action plan | The step of creating one or several strategies to respond to what is asked. It refers to the use of already known strategies, coming from other problems solved, use of properties, simplification of the original problem in easier and less time-consuming parts, determination of tasks and division of responsibilities. | Have you found a similar problem? Do you know any problem related to this? Do you know any theorem that might be useful? Is this a problem related to another that has already been resolved? Could you use it? Could you use your result? Could you use your method? Do you think it would be necessary to introduce an auxiliary element to be able to use it? Could you state the problem differently? |
| Executing the plan | Step in which the planning is carried out, fulfilling or not all the phases, modifying those elements that stand as obstacles to the solution of the problem and verifying or refuting the hypotheses of the plan, replanning, until finding the desired solution. | Have you written your action plan yet?<br>Are the planned paths helping in formulating the problem?<br>What are the obstacles?<br>Do you need replanning? |
| Retrospective view, evaluation of the plan. | Step of action monitoring. It is important to emphasise two aspects: the evaluation of the effectiveness and efficiency of the plan in function of the comparison made with other plans presented to solve the same problem; validation of the solution found, generalisation as a tool to elaborate other strategies to be used in another problem. | Can you check the result? Can you check the reasoning? Can you get the result differently?<br>Can you employ the result or method in any other problem? |

Source: Kaiber and Groenwald (2008, p. 236)

The definition of the problem used in this research is that of Polya (1995), where the problem arises when you look for ways/means to achieve an immediate goal, occupying most of the thinking part with incessant searches to find a satisfactory solution. According to this author, a problem has three characteristics: there must be someone willing to solve it; there must be an initial state and a final state to be achieved and a possible impediment in the passage from one state to the other must be explored.

Echeverría and Pozo (1998) cite that:

> Teaching problem solving is not only about equipping students with effective skills and strategies, but also about creating the habit and attitude of facing learning as a problem for which an answer must be found. It is not a question of teaching only to solve problems, but also of teaching to *propose* problems for oneself, to turn reality into a problem that deserves to be questioned and studied [...] The real ultimate goal of problem-solving learning is to make the student acquire the habit of proposing problems and solving them as a way of learning (ECHEVERRÍA in POZO, 1998, p. 14-15).

NCTM (2000) states that:

> Solving problems is not only a goal of learning mathematics, but also an important way of doing it. Problem solving is an integral part of all math learning and therefore should not be just an isolated part of the math program. Problem solving in mathematics must involve all five content areas described in the NCTM Standards. Good problems integrate multiple topics and will involve significant mathematics (NCTM, 2000, p. 52).

Research works point out that the main function of problem solving should be to develop students' mathematical understanding and that students understand or do not understand certain concepts or contents. Usually this is manifested when they solve problems.

Thus, working the development of algebraic thinking through problem solving allows the student to understand the applications of algebra, developing algebraic concepts. For NCTM (2000), ensuring that students have the opportunity to engage with high-level thinking, teachers must select and implement daily tasks that stimulate thinking and problem solving.

In this sense, this research seeks to identify the performance of the students of the 9th grade of elementary school in solving problems with the contents of first degree equations and systems of first degree equations.

## The research

The general goal of this research was, as stated before, to examine whether 9th-graders of an elementary state school based in the municipality of Porto Alegre, state of Rio Grande do Sul, have developed the competence to solve problems involving algebraic thinking in the contents of first degree equations and systems of first degree equations.

The 9th grade of elementary school constitutes a step of completion of the construction of the students' competences, including elementary algebraic reasoning, which justifies the group as an eligible corpus for this study. To apply the adaptive tests in school we decided that, as prerequisite for the subjects covered in the tests, the students would have to have worked with equations in the 7th grade, and with algebraic contents, including systems of first degree equations, in the 8th grade.

In the SIENA system, a question bank was created with 450 questions, divided into three levels: easy, medium and difficult, all using algebraic thinking, which generates adaptive tests. In the elaboration of the questions, the documents originated from the NCTM, APM[8], MEC[9] (PCN, ENEM and the textbooks of the PNLD[10] (2013) adopted in basic education) were used as reference.

SIENA, according to Groenwald and Moreno (2006), allows the teacher to enquire into the level of previous knowledge of each student and allows a planning of teaching according to the reality of the students.

After applying the adaptive tests to the 30 students (17 girls and 13 boys, aged 14 to 17 years), both the SIENA database and the records of the development of the questions they asked were scrutinized. During the tests, one student was transferred. The class was composed of students who had attended practically all elementary school in that same institution.

The SIENA system used in this research is presented below.

## Integrated system of teaching and learning (SIENA)

SIENA[11] is an intelligent system capable of communicating information about students' knowledge on a specific subject. It is intended to aid in the process of retrieving mathematical contents using a combination of conceptual maps and adaptive tests (Groenwald & Moreno, 2006). SIENA generates an individualised map of the students' difficulties, which will be linked to didactic sequences, and will serve to recover the difficulties that each student presents in the

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[8] APM – Associação Portuguesa de Matemática (Portuguese Association of Mathematics)
[9] MEC – Ministério da Educação e Cultura (Ministry of Education and Culture)
[10] PNLD – Programa Nacional do Livro e do Material Didático (National Programme of Textbook)
[11] The text on the SIENA system, used in this project, is a standard text developed by the Group of Mathematics Education Curriculum Studies (GECEM) of ULBRA and the Group of Educational Technologies of the University of La Laguna in Tenerife, Spain, developers of SIENA.

content evolved.

It was developed through a variation of the traditional concept maps (Novak & Gowin, 1988), being denominated *Pedagogical Concept Instructional Graph* - PCIG, that allows the planning of teaching and learning of a specific theme. The graph does not organise the concepts according to arbitrary relations, the concepts are placed according to the logical order in which they must be presented to the student. Therefore, the graph must be developed according to relations of the type "concept A should be taught before concept B", starting with the nodes (topics in the graph) of the previous concepts, following to the fundamental concepts, until reaching the objective nodes.

Each concept of the graph is linked to an adaptive test that generates the individualised map of the student's difficulties. A computerised adaptive test is administered by the computer, which seeks to adjust the test questions to the skill level of each examinee. According to Costa (2009), an adaptive computerised test seeks to find an optimal test for each student, for which the individual's proficiency is estimated interactively during the administration of the test and, therefore, only those items that efficiently measure the proficiency of the examinee are selected. Its purpose is to administer questions from a bank of previously calibrated questions that match the level of the examinee's skills. As each question presented to an individual is suited to his or her skills, no test question is irrelevant (Sands & Waters, 1997). Unlike paper and pen tests, each student receives a test with different questions and varied sizes, producing a more accurate measurement of the proficiency and with a reduction of the test size by around 50% (Wainer, 2000).

To estimate the student's knowledge in each concept of the graph, SIENA implements a Bayesian network between the concepts involved in this node of the graph, and the multiple-choice questions created for these concepts are divided into several levels of difficulty. The adaptive test adapts to the student's knowledge by choosing a question of equal or greater difficulty, if the previous question was answered correctly, and equal or less difficulty, if the question was answered incorrectly (wrong answer).

The system has two options of use: the first one is for the student to study the contents of the nodes of the PCIG and to perform the test, to check their knowledge about certain contents; the second option allows the student to take the test and study the concepts in which they presented difficulties, being possible an individualised recovery of the contents in which he could not surpass the average stipulated as necessary to advance. All the nodes of the PCIG are linked to a didactic sequence that enables the student to study the concepts or perform the recovery of the nodes in which they present difficulties.

In this investigation, the SIENA adaptive tests were used to analyse the data collected in the experiment with the students, as already presented. For this, the following actions were necessary: the construction of the graph with the skills to be evaluated; the construction of the question bank for the adaptive tests of each topic of the graph, with 45 questions in each topic.

## Research environment

The graph of the topics to be worked was composed of 10 topics, divided as follows: Mathematical language; The question to the problem; Retrieving data from a problem; Simple 1st degree equation; Elaborated first grade equation; Simple problems; Elaborated problems; Identification of the system of equations; Solving systems of simple equations and Solving systems of elaborated equations.

The adaptive tests are designed to organize the questions of the SIENA database. To compose this database, it is necessary to register the questions for each topic, aiming to evaluate the individual knowledge degree of each student. The questions are multiple choice and should define: the degree of difficulty (easy, medium or difficult); identify the true response; the possibility of answering the question considering random checking; the time for the student to answer the question (in seconds) and the prior knowledge of the student for that topic (in the teacher's view). The student will be considered apt when he or she achieves 0.6 in the interval of [0,1].

Below, we show examples of questions that compose the questions database of the adaptive tests, with the objective and an example of questions for each level of difficulty (Frame 7).

Frame 7 - Adaptive tests

<table>
<tr><td colspan="3">Topic: Mathematical language</td></tr>
<tr><td colspan="3">The aim of this topic was to analyse if the student can transpose from the mother tongue into the mathematical language, demonstrating their comprehension through choosing the equation that solves the proposed activity. At the easy level, the four basic operations were used, as well as trivial notions such as double, half, etc. At the middle level, the same notions were used, using language that requires greater difficulty than at the easy level, adding operations involving unknowns. At the difficult level, besides previous mathematical skills, we also added knowledge involving operations with unknowns, consecutive numbers, even numbers, odd numbers, etc.</td></tr>
<tr><td>Easy Level: Which expression represents a number minus thirty-six?</td><td>Middle Level: Five times a number plus its third part minus ten may be represented by the expression:</td><td>Difficult Level: Which expression is twice the number added to triple the half of its consecutive?</td></tr>
</table>

<table>
<tr><td colspan="3">Topic: Elaborated first grade equation</td></tr>
<tr><td colspan="3">The objective of this topic was to verify if the students could interpret the statement, assemble the equations and solve more elaborated 1st grade equations using mathematical notions beyond the four operations. The difficulty levels of this topic are centred on the interpretation of the statements and the resolution of operations involving the unknowns on both sides of equality.</td></tr>
<tr><td>Easy Level: The sum of two consecutive integers is -31. What are these numbers?</td><td>Middle Level: The sum of a number with its consecutive is 25. What are these numbers?</td><td>Difficulty Level: The sum of two numbers is 76. It is known that the larger number is 6 units larger than the other. What are these numbers?</td></tr>
<tr><td colspan="3">Topic: Elaborated problems</td></tr>
<tr><td colspan="3">The objective of this topic was to verify if the students have the ability to transcribe from the mother language into the mathematical language, solving the problem situations involving algebraic knowledge, solving 1st grade equations with any kind of mathematical operations.</td></tr>
<tr><td colspan="3">The age of a father is three times his son’s age. Calculate these ages, knowing that, together, they are 60 years old.</td></tr>
<tr><td colspan="3">A school received 1,350 enrolments for the 7th, 8th and 9th grades in 2015. There were 420 students for the 7th grade and, for the 8th grade, twice the number of enrolments for the 9th grade. How many students enrolled in each grade?</td></tr>
<tr><td colspan="3">Silvio rented a car at Agency X for R$ 280.00, plus R$ 3.00 per km of use. Pedro rented, in Agency Y, for R$ 400.00, plus R$ 1.00 per km of use. For them to spend the same, the distance travelled by them should be:</td></tr>
<tr><td colspan="3">Topic: Systems of equations</td></tr>
<tr><td colspan="3">The objective of this topic was to verify if the students were able to identify the equations that form the systems of equations with one or two unknowns, according to the presented problem. The difficulty level was determined according to the type of problem we have (type of interpretation) and the operations involved.</td></tr>
<tr><td>Easy Level: In a parking lot there are cars and motorcycles, totalling 78. The number of cars is 5 times that of motorbikes. How many motorbikes are there in the parking lot?</td><td>Middle level: In the first Regional Games of the Central-West Region, the women's athletics teams of Marília and Araçatuba totalled 377 points. Marília scored 31 points more than Araçatuba. How many points did each team score?</td><td>Difficult Level: (UNAQ - 2011) - One day, a cafeteria sold 16 cups of orange juice and 14 cups of pineapple juice, receiving a total of R$ 67.00. A person bought a glass of juice of each type, paying, in total, R$ 4.50. So, the difference between the price of the juice glasses is:</td></tr>
<tr><td colspan="3">Topic: Solving systems of elaborated equations</td></tr>
<tr><td colspan="3">The objective of this topic was to solve problem-situations involving systems of first degree equations, solving through systems with two unknowns, either by the process of addition or isolation of the unknown. The level of difficulty is directly related to: 1st) the interpretation of the problem, that is, to the amount of text used; 2nd) the actual solution of the system of equations, since it involves two unknowns.</td></tr>
<tr><td colspan="3">Easy Level: Ari and Rui have, together, R$ 840.00. Ari's amount equals ¾ of Rui's amount. Therefore, Rui has:</td></tr>
<tr><td colspan="3">Middle Level: I have 20 banknotes, some of R$ 5.00 and others of R$ 10.00. The total amount of bills is R$ 165.00. How many bills of R$ 5.00 and how many of R$ 10.00 do I have?</td></tr>
<tr><td colspan="3">Difficult Level: (CTSB - 2009) - Two friends went to the supermarket together to buy wine. One of them bought 3 bottles of wine A and 2 of wine B, paying in total R$ 79.00. The other bought 5 bottles of wine A and 1 of wine B, paying in total R$ 92.00. One can conclude that 1 bottle of wine A costs, in relation to 1 bottle of wine B.</td></tr>
</table>

Source: The research

## Research analysis

Students will be identified as Pal XX, where Pal is the identification of algebraic thinking on the platform and XX represents the number of each student.

In the *Mathematical Language* topic, 100% of the students reached 0.6, necessary score to have their adaptive test considered satisfactory. No student scored below 0.9313.

Student PAL25 solved a test with 10 questions, getting them all right. In the *difficulty* column, the SIENA system provided the first question of level 0.4, considered difficult and, as this student was getting the answers right, it always maintained the level of difficulty.

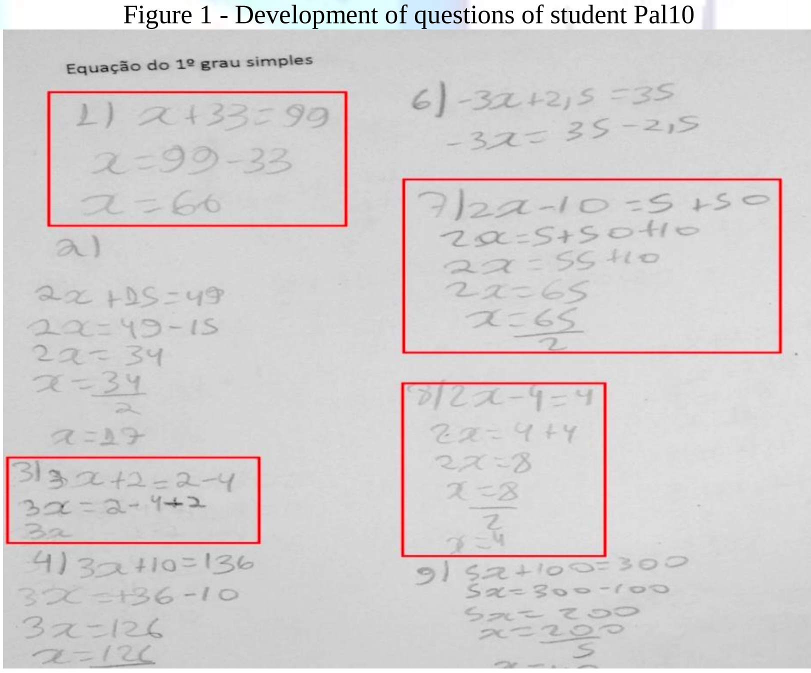
In total, 366 questions were solved in this topic, distributed into the three levels of difficulty, Here, the students failed in 47% of the questions answered.

We can observe that the students dominate the representation of this language, but 62% out of the students who got the questions wrong missed difficult-level questions, showing that they do not master the ones that need deeper text interpretation.

In the topic *Simple first degree equation*, 73% reached the minimum score of 0.6, of that percentage 50% got a score higher than 0.99. There were 385 questions on this topic, distributed into 130 difficult, 142 medium and 113 easy questions. All students received at least a difficult level question, even those who could not reach 0.6. This reveals that they have moved from the easy level to the middle level, being able to interpret the mathematical language, identify the problem question and solve the simple 1st degree equations formed.

After analysing student Pal10's field diary, we can assume that the incorrect questions were due to misinterpretation (Figure 1).

Figure 1 - Development of questions of student Pal10




Source: The research

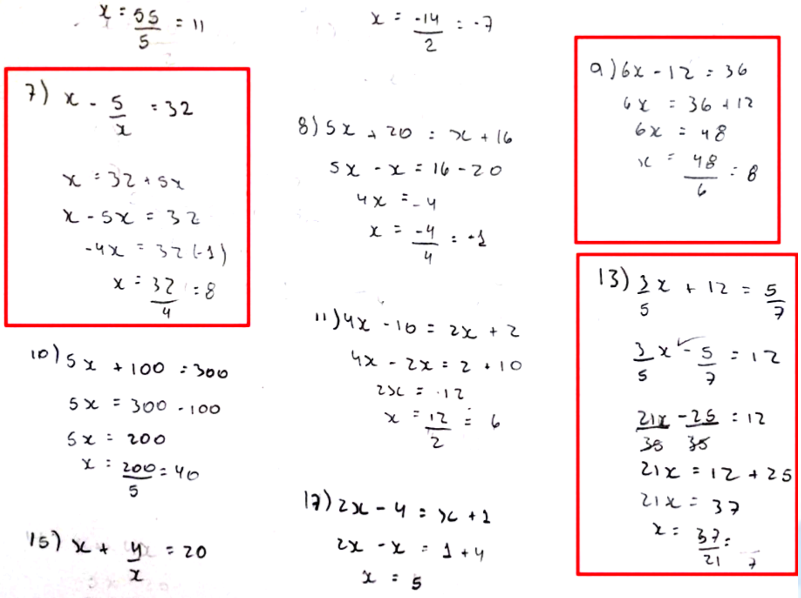
In exercise 1, the statement said: "triple a number," but Pal10 wrote only the number, missing its development, and, consequently, the transcription to the mathematical language. In activity 7, where the problem stated: "Twice a number minus ten equals its half plus fifty", we observed that the student found correctly the double of a number minus ten, but in the part of its half, the student put five, interpreting that it was half of the ten, that is, his interpretation of the problem was erroneous.

In this topic, the resolution of the equations was successful, the wrong questions were related to interpretation and to the assembly of the equations. Therefore, by the percentage of correct answers and analysis of the developments in this topic, the objective was reached.

In the topic *Elaborated first degree equation*, 15 students achieved scores above 0.6. Four students had their tests invalidated for extrapolating the maximum time allowed in all their attempts and one student was transferred from the school. One student had a maximum score, 1.0, and one student scored 0.0. Students who were below 0.6 were well below this value. Of the total of 321 questions, 38% were difficult, showing that students transposed the other levels, even though they had a 50.46% error rate. Of the questions answered wrong, 48% are at this level of difficulty. For students to have this error rate on difficult issues, they must pass through the other two levels. This means that within the levels of difficulty that are centred on the interpretation of the statements and the resolution of operations involving the unknowns on both sides of equality, they have had difficulties concerning deeper interpretation.

Student Pal01 answered 20 questions but could not get a 0.6. We observed that the system started testing him with a difficult level question, and, as he missed the question, the system lowered the level, raising it as soon as he got it correctly. Figure 2 will show examples of this student for analysis.

Figure 2 – Student Pal01's field diary



Source: The research

Observe these three resolutions. Student Pal01 understood that problem 6, in the system (which he represented in his resolution sheet as exercise 7), was a subtraction of two values, where one refers to the number and the other to the fifth part, but he did not know how to represent that fifth part, because, instead of representing it by $\frac{x}{5}$ he represented it by $\frac{5}{x}$, which led him to the wrong resolution.

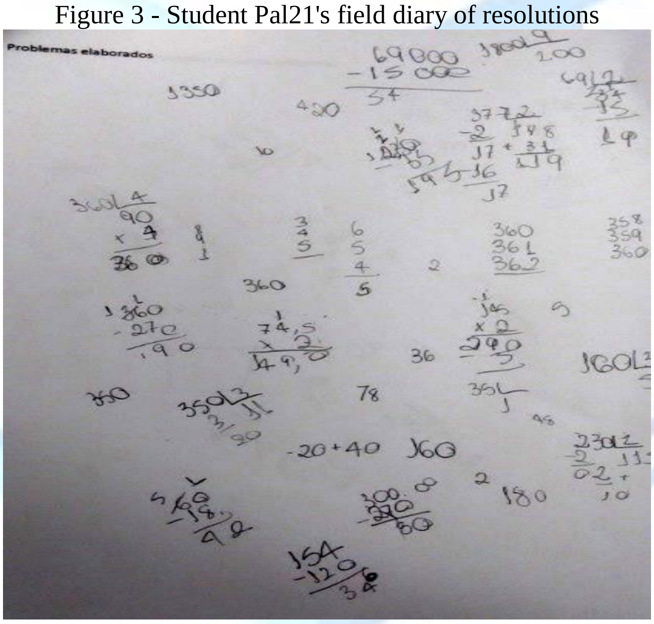
A similar fact occurred in question 13 of his resolution, in which the statement said the following: " $\frac{3}{5}$ of a number increased by 12 is equal to $\frac{5}{7}$ of that number." Note that the student wrote $\frac{3}{5}x + 12 = \frac{5}{7}$, forgetting the part $\frac{5}{7}$ of that number once more, getting wrong the resolution of the system question once more.

Of the 15 students who reached the minimum grade, 8 answered 10 questions; for the remainder, a greater number of questions were required to be answered. The objective of this topic was to verify whether the students could interpret the statement, assemble the equations and solve more elaborated 1st degree equations using mathematical notions beyond the four operations. The students demonstrated difficulty in interpreting the statements, since the difficulty levels were centred in the interpretation of the statements and later resolution of the problem.

In the analysis of the topic *Simple problems*, the students answered 360 questions, divided into 98 easy, 105 intermediate and 157 difficult-level questions: 15 students got the minimum score and 12 students almost failed the test. No students scored 1.0 and those who did not score 0.6 got much lower scores, with the "nearest" being 0.31.

In the topic *Elaborated Problems*, a total of 328 questions were distributed in the three levels of difficulty, being 134 in the easy level, 103 in the average and 92 in the difficult level. In this topic, it was observed that the students showed difficulty to interprete, sometimes asking questions about interpretation during the test application. Thus, only 48% of them managed to reach the minimum score of 0.6. Six students had their tests invalidated because they could not answer the questions presented. We also had 9 students who scored less than 0.1, while two scored the maximun, 1.0. Only 11 out of the students who had managed to reach 0.6 in the simple problems topic hit it again in the elaborate problems. They demonstrated difficulties in the resolutions, sometimes not using algebraic artifacts as shown in figure 3, which includes the students' field diary of resolutions.

Figure 3 - Student Pal21's field diary of resolutions




Source: The research

We can note that the student, in achieving a more accurate interpretation, does not use many algebraic strategies in his resolution. This is one of the students who scored above 0.6 in the adaptive test.

In the topic *Systems of equations*, we observe a total of 318 questions answered, being 121 questions at the easy level, 98 at the medium and 99 at the difficult level. Out of this total, almost 54% of the questions were answered erroneously. Fourteen students scored above 0.6 and six students had their tests invalidated for extrapolating time and/or failing to respond to problems.

The difficulty levels in this topic were determined by the type of interpretation requested and the operations involved. The questions in which the students made more mistakes were those of easy level, showing difficulty to interpret the statements. In the distribution of the correct answers' checkings, it was observed that the questions that the students got correctly were the ones of medium level, but that there was not great difference between the levels.

In the analysis of the topic *Solving simple system of 1st degree equations*, a total of 336 questions were answered, distributed in the three difficulty levels, 130 questions at the easy level, 106 at the medium level and 100 at the difficult level. From this total, students failed in 56% of the answers. During the development of the test, only 3 out of those students who concluded it failed to reach the difficult level questions; the others were able to develop their tests up to that level. It is noteworthy that 59% of the students participating in the test failed to reach 0.6 and only 4 had a maximum score.

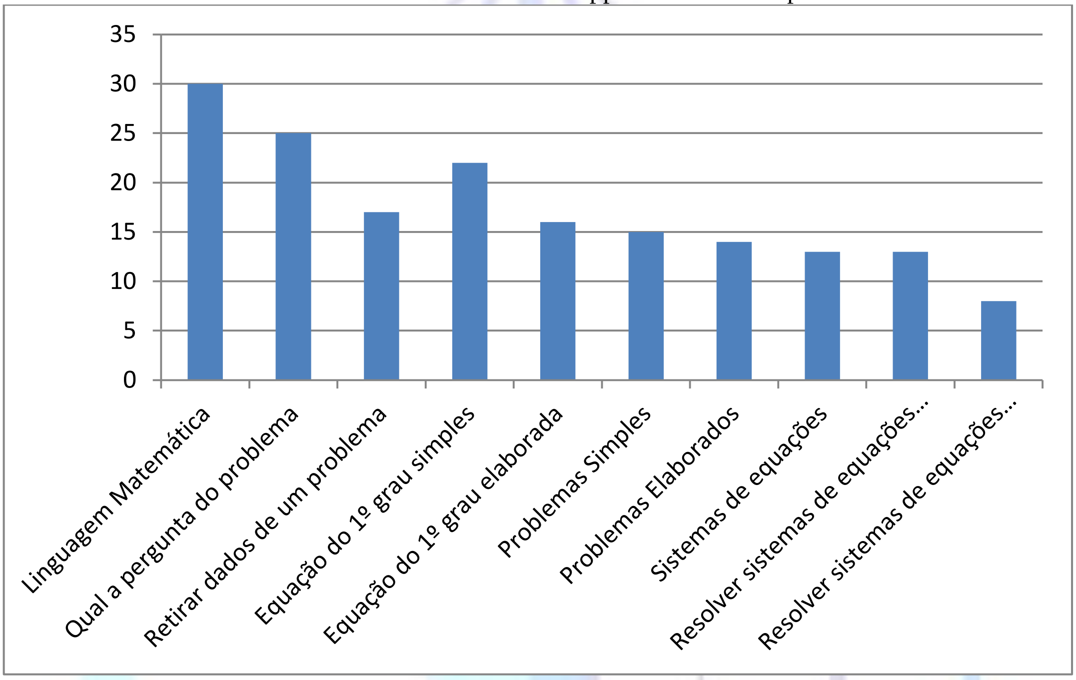
In the analysis of the topic *Solving the first-degree equation system*, only 24% of the students reached the objective of solving problem situations involving systems of equations of the first degree with two unknowns. Only 8 students scored above 0.6 on this topic and 10 invalidated their tests. Also, 9 students had scores below 0.01. 275 questions were answered, 168 of which were answered incorrectly. The high percentage of errors at the easy level demonstrates that students have shown great difficulty in reaching the other levels. Of the 57 answers given to the difficult level questions, 44 were answered by the 8 students who scored between 0.6 and 1.0. Of the 9 students who did not score 0.6, and who had the tests validated, only 3 did not receive questions of that level to answer. In this topic, what defines the difficulty is the interpretation of the problem, that is, the type of text used and the actual resolution of the equation system, in order to find the correct answer.

## Final considerations

Students performed relatively well on topics requiring basic interpretation of operations involving algebraic components. The students, within the steps highlighted by Polya (1995), managed to realize the problems. In the second step, establishing a plan for the resolution, the students also managed to reach their objectives, because it was observed that, even if they got some questions wrong, they used algebraic schemes to solve the problems. During the execution of the problems, they had difficulties with issues where further reading was necessary.

The topics in which the students excelled were *Mathematical language* and *Simple 1st degree equation*. Chart 1 shows a graph of the number of students who passed each topic, that is, those who scored above 0.6.

Chart 1 - Number of students approved in each topic




Source: The research

It is observed that the topic with the lowest number of students that reached the objective was *Solving system of elaborated equations*, where only 26.67% achieved approval. The students presented difficulties in assembling and analysing the systems, thus compromising all the performance of this topic, not being able to organise the system of first degree equations.

One negative point observed was that some of them, when confronted with questions that had very long statements, were discouraged and sometimes did not complete the questions either because they did not understand the statement or because they could not formalise a reasoning.

The students, in general lines, obtained a good performance in the adaptive tests, reaching the proposed objectives, because they succeeded in more than 50% of the topics used in this research. In some situations where they showed greater difficulty, they needed a more accurate algebraic basis for solving problems, always trying to bring them to arithmetic, rather than to think algebraically.

It is possible to affirm that the students of the 9th grade of elementary school of the state education network of RS have developed the problem-solving competence that involve algebraic knowledge. But it is observed that the algebraic knowledge of these students needs a greater emphasis in the domain of algebraic language when reading longer texts, and when solving a system of equations.

The students found the tests very interesting but claimed to have found it difficult to interpret the statements. However, they managed to accomplish most of them. They also put as a positive point of the tests performed the possibility of perceiving their small or big mistakes and try to correct them.

The topic in which they found the greatest difficulty was the one that involved systems of equations, since some students were able to perform the assembly of the system and made mistakes when solving it. The topic they were more comfortable with was the one of *Mathematical language*, in which they, according to the problem, would indicate the alternative that contained the numerical expression corresponding to the problem.

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