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Perspective Teaching for High School Students in a Dynamical Geometry Environment

O Ensino de Perspectiva para Alunos do Ensino Médio num Ambiente de Geometria Dinâmica

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ABSTRACT

It is presented results of a proposal using perspective ideas for Geometry's teaching, with High School students. Based on historical development of bidimensional representations of spatial figures techniques and supported by Parzysz's and Gutiérrez's studies on those representations, activities were developed, to explore techniques for constructing images in perspective, in a Dynamic Geometry environment. Data analysis shows that using a software may provoke interest on geometric constructions and Geometry, and although there was restricted apprenticeship of concepts that depend on previous ones, some were more frequently assimilated, which shows that a work of this nature may promote learning in Geometry.

KEYWORDS: Perspective, Dynamic Geometry, Geometry.

RESUMO

Apresentam-se resultados de uma proposta de ensino de Geometria para alunos do Ensino Médio, ampliados a partir do trabalho apresentado no VII SIPEM. A partir do desenvolvimento histórico de técnicas para a representação plana de figuras espaciais, procurou-se suporte teórico para essas representações em Parzysz e Gutiérrez para elaborar atividades que exploram técnicas de construção de imagens em perspectiva com Geometria Dinâmica. A análise dos dados evidencia que usar um software pode despertar interesse por construções geométricas e Geometria e embora mostre limitações no aprendizado de conceitos que dependem de conhecimentos prévios, alguns foram assimilados pela maioria, mostrando que um trabalho dessa natureza pode promover aprendizagem em Geometria.

PALAVRAS-CHAVE: Perspectiva, Geometria Dinâmica, Geometria de Posição.

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Introduction

The origins of perspective techniques for plane representation of spatial figures date back to the 15th century and have been present in many works of art ever since. The aim of this work was to organise a set of activities that would provide High School students with an initiation to the geometric ideas related to perspective techniques, considering a historical route of their development. As the images in perspective can be identified with plane representations of three-dimensional figures, we consider that knowing aspects and principles adopted in these representations may collaborate for the development of visualisation skills.

The Curriculum Parameters for Secondary Education - PCNEM (BRASIL, 2015) refer to the relationships between visualisation and representation.

> [...] visualisation, drawing, logical argument and application skills in the search for solutions to problems can be developed with an adequate geometry work, so that students can use geometric shapes and properties to represent and visualise parts of the world around them (BRASIL, 2015, parte III, p. 44).

According to the National Common Core Curriculum - BNCC (BRASIL, 2017),

> To establish relationships between orthogonal views and representations in perspective of spatial geometrical figures and objects of the physical world and apply this knowledge in situations related to the world of work. (BRASIL, 2017, p. 565).

In analysing five collections of elementary education mathematics textbooks approved by the Didactic Book National Program (BRASIL, 2014) before BNCC, we only found reference to perspective techniques in one 9^{th} grade volume. In chapter "Como aprender a desenhar em 3D/How to Learn to Draw in 3D", the authors analyse paintings that highlight the notion of depth and explain the construction of a square or a rectangle with the conical perspective based on the concepts of vanishing point and horizon line. Considering the recommendations mentioned above and the lack of didactic material for geometry teaching in basic education, we have proposed to investigate whether aspects related to perspective may contribute to the development of visualisation skills in a group of High School students, while providing them with an opportunity to practice geometric properties they have studied in the geometry of position field. This article brings some of the results obtained with this research, which aimed at introducing and presenting to a group of High School students some properties of the projective geometry through initial ideas of perspective and a set of activities with resources that allow the transition from a static approach (figures, paintings, paper and pencil) to a dynamic context (provided by the use of dynamic geometry software). The work with the aid of the software allowed us to rescue the context of Dürer's practice[4] to copy the image to be represented on canvas, by fixing the position of the person who is drawing. From the reproduction of this practice in the environment of dynamic geometry, elements of the geometry of position can enable us to understand the elements and the first rules of representation formulated by Alberti (1991). In this way, we revisit the techniques of practice and theory of representation of the 15th century with the aid of technology, in an activity developed in the classroom.

General research considerations

The introductory historical ideas on the techniques of plane representation of spatial figures were based on Alberti's work (1991) and Dürer's technique of representation. Also relevant were the works by Souza (2010), Miranda (2006), Kodama (2006) and Meneguzzi (2009), among others. Souza's work (2010) investigated the use of images for the development of visualisation skills. Miranda (2006) studied the plane sections of a cube that can be visualised through descriptive geometry, supported by a dynamic geometry software, identifying relationships between a figure and its representations in spatial geometry and descriptive geometry. Kodama (2006) studied the Cavaliere perspective and Meneguzzi (2009) explored a historical sequence, which led us to Dürer, creator of the perspective machines - the perspectographers -, combining the history of the perspective with the knowledge used in drawing techniques.

Our intervention was guided by "Design Experiment" Methodology ideas (COBB et al., 2003), due to its evaluative and reflexive nature, since understanding the students' comprehension is an essential part of our objectives. Evaluating what is taught in one stage and then resuming it in the next stage, based on the results obtained and so on, as suggested by Cobb et al. has contributed to this research.

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[4] http://www.uh.edu/engines/epi138.htm

We have tried, as in the original research, to answer the following question, among others: "Does the use of a dynamic geometry software to work with projections of figures make it easier the understanding of those projection techniques?"

For data collection, we elaborated a set of activities that were carried out in three meetings with ten students of the 2nd grade of high school of the state public network of Ceará. In the first meeting, we discussed historical aspects of the evolution of the perspective techniques and the geometric concepts that support them, such as parallelism, midpoints and diagonals, from works of art from different periods. Then, we introduce the notions of "vanishing point" and "horizon line", with the support of elements of geometry of position, from the re-creation of the strategies of the practice of representation and retracing, in the environment of dynamic geometry, a path which goes from this practice to the rules established by Alberti in "Della Pintura" (ALBERTI, 1991). In the second meeting, after the perspective construction of a parallelogram, the participants used paper, pencil and ruler to construct and explore properties of the perspective representation of parallelograms. In the third and final meeting, we presented and discussed other examples of basic construction strategies, reproduced in a dynamic geometry environment. The participants answered some questions about the influence of the use of the computational resource in the understanding of the concepts introduced by the presented models. In this article we will bring a part of the activities held in the meetings to analysis.

Data analysis was carried out in the light of studies by Parzysz (1998) and Gutiérrez (1998) on plane representations of spatial figures.

Parzysz (1998) investigates the loss of information in representations, drawing a parallel between knowledge and perception and the conflict between the *pole of what is seen* and the *pole of what is known*. According to Parzysz, the pole of the seen is characterised by the visual representation of an environment or object, and can only describe it by the elements that are presented there; the pole of the known refers to what is known of an object or environment, not linked to what is represented, but to characteristics and properties. For example, in a perspective representation, we "see" that the straight lines representing the floor are not parallel - pole of the seen - if we limit ourselves to what the image reveals to us, but we "know" that they represent parallel lines - pole of the known - if we interpret it by the properties we know of the environment and not only by what is seen in the representation.

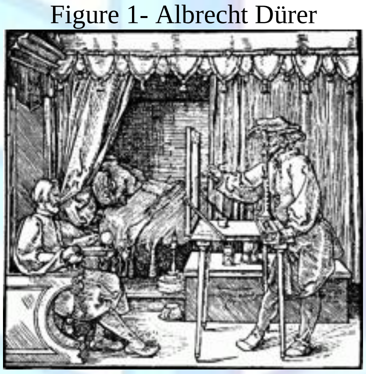
Gutierrez (1998) proposes a set of skills necessary to make or interpret plane representations of spatial figures and, as Parzysz (1998), considers that plane representations can generate difficulties he summarises in two: interpreting the plane representation to understand the original three-dimensional figure, which Parzysz considers decoding the representation; and understanding the three-dimensional object in order to construct the plane representation, that is, codifying the figure, according to Parzysz.

We think that the use of a dynamic geometry software can help to understand the perspective and the differences and similarities between real environments and representation environments. Our activities sought to show how plane representations of three-dimensional environments and objects work "in practice", aiming to generate knowledge that serves to understand real-life situations.

The first perspective techniques

The first perspective techniques emerged in Italy in the 15th century, but since the 13th century artists already pursued to create images closer to reality, through resources such as colour and position of the elements.

Figure 1- Albrecht Dürer

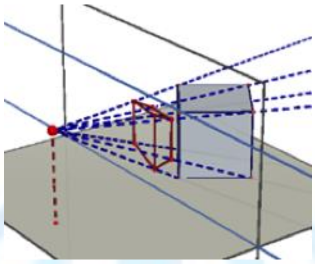


Source: http://venetianred.net/2010/03/06/hans-holbein-tricks-of-the-trade/

Figure 1 reproduces the practice used by the German artist Albrecht Dürer (1471-1528) to sketch the representation of an image on the canvas; this practice was discussed by the participants when analysing the strategies of representation contained in a set of works of art from different epochs. As shown in Figure 1, the painter, with his head held in a fixed support, reproduced the image on the transparent screen placed between him and the scene to be painted.

Figure 2 shows how, using the software, we can reproduce the practice of representations made with Dürer's equipment. Let us a cube resting on a horizontal plane and its projection on a vertical plane.

Figure 2 - Plane representation using the software



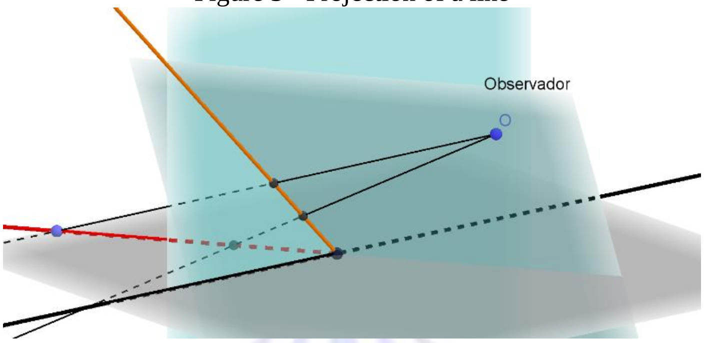
Source: Personal Collection

From a fixed point, the "observer", semi-straight lines originating from that point are drawn, passing through points chosen in the figure to be represented; the points of intersection of the semi-straight lines with the vertical plane of projection are marked, forming the plane representation of the three-dimensional figure. In Figure 2, the polygon formed by the segments in the vertical plane, called the "projection plane", represents the parts the observer sees, so it shows only three of the six faces of the cube to be represented, which leans on a horizontal plane we will call "real space", and contained in one of the quadrants determined by this plane and the projection plane. The intersection of the two planes is a straight line, called the "ground line" we will name *t*.

We used the software to investigate, with the students, some important properties that become evident when we perform the parallel line projections contained in the horizontal plane.

We first observed that, by fixing the point *O*, which represents the observer, the projection of a line contained on the horizontal plane can be obtained by the projections of two distinct points of the original line, or as the line of intersection of the plane determined by that line and the point *O*, which we call auxiliary plane of projection, and the vertical plane of projection, as in Figure 3. It is worth noting that the auxiliary planes of projection form a beam of planes whose intersection is a line passing through *O*.

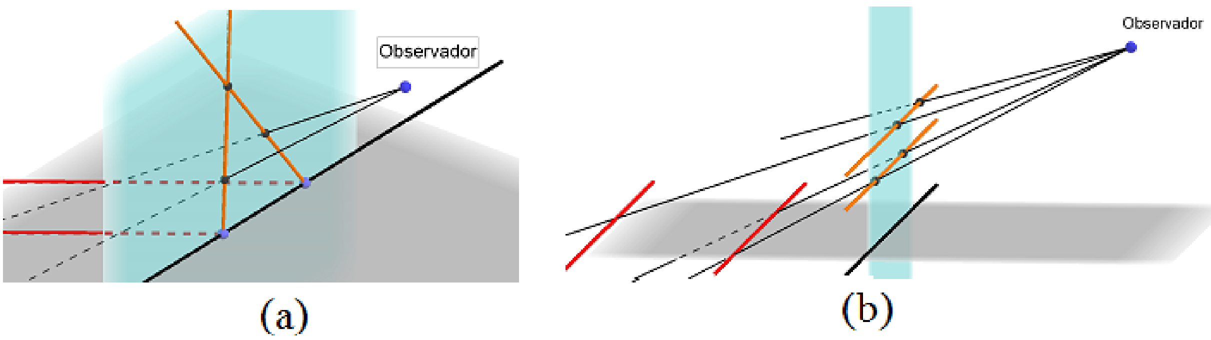
Figure 3 - Projection of a line



Source: Personal Collection

Then, with the dynamicity of the software, we could observe that given two parallel lines in the real plane, their projections are either competing lines, if they intercept the ground line (Figure 4, (a)), or lines that are parallel to the ground line, as in Figure 4, (b).

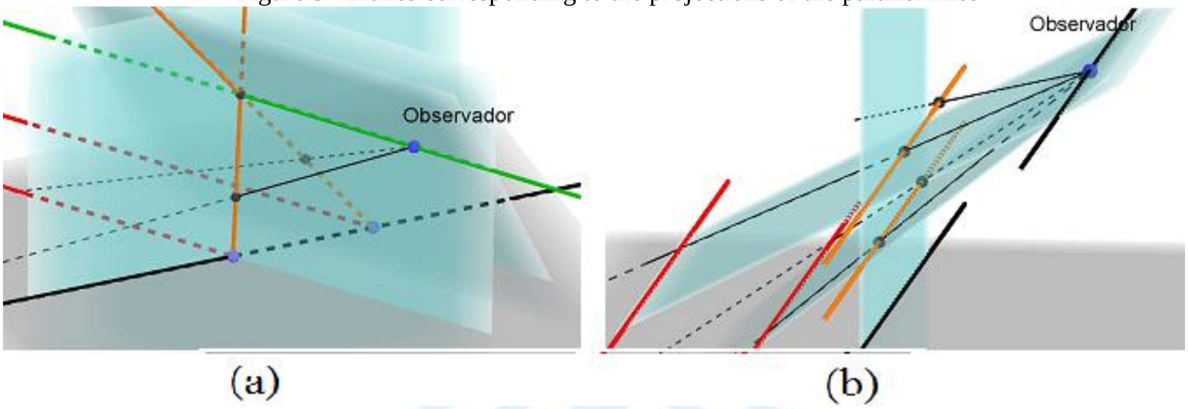
Figure 4 - Parallel line projections



Source: Personal Collection

The software allowed to visualise the auxiliary planes that generate the competing (Figure 5 (a)) or parallel (Figure 5 (b)) projections. In the first case, the auxiliary planes form a beam of planes whose intersection is a line that is parallel to the real lines, and that passes through the point *O* and the point of competition of the projected lines. This point is the so-called "vanishing point". In the second case, the planes form a beam whose intersection is a line passing through *O*, parallel to the vertical plane of projection. If the original lines are perpendicular to the ground line, the line of intersection of the auxiliary planes will be perpendicular to the plane of projection.

Figure 5 - Planes corresponding to the projections of the parallel lines



Source: Personal Collection

After analysing the possible situations for the projection of two parallel lines, the possibilities for the projection of a parallelogram were discussed (Figures 6). In Figure 6 (a), we have the projection of a square with one of the sides parallel to the ground line; in (b), the projection of a parallelogram with one of the sides parallel to the ground line. In (c), the projection of a parallelogram whose sides are not parallel to the ground line. In this case, we have two competing lines at point *O*, each parallel to the straight line that supports the sides of the parallelogram. These lines form a plane that intersects the plane of projection along a line passing through the vanishing points *P* and *Q* (Figure 7). This line is called the "horizon line".

Figure 6 - Possibilities for the projection of a parallelogram

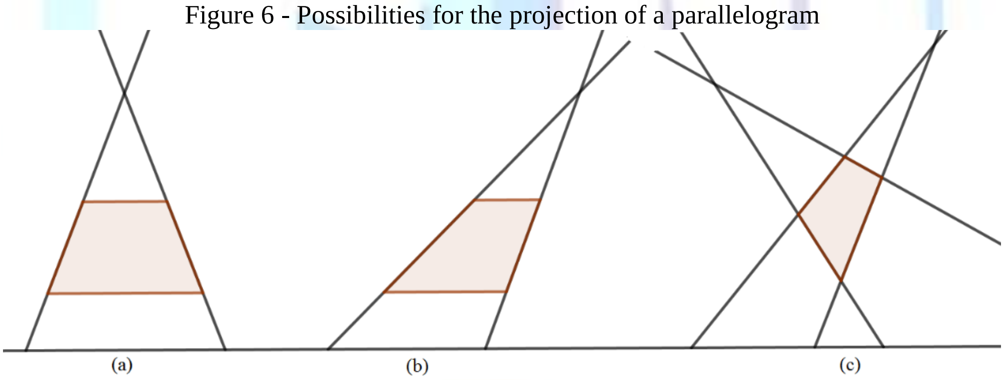


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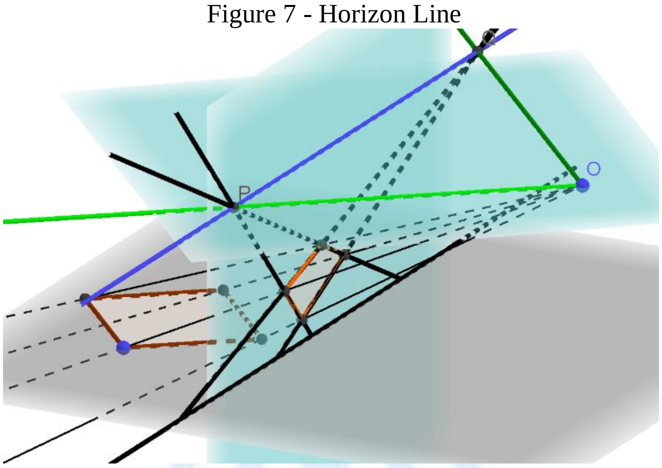
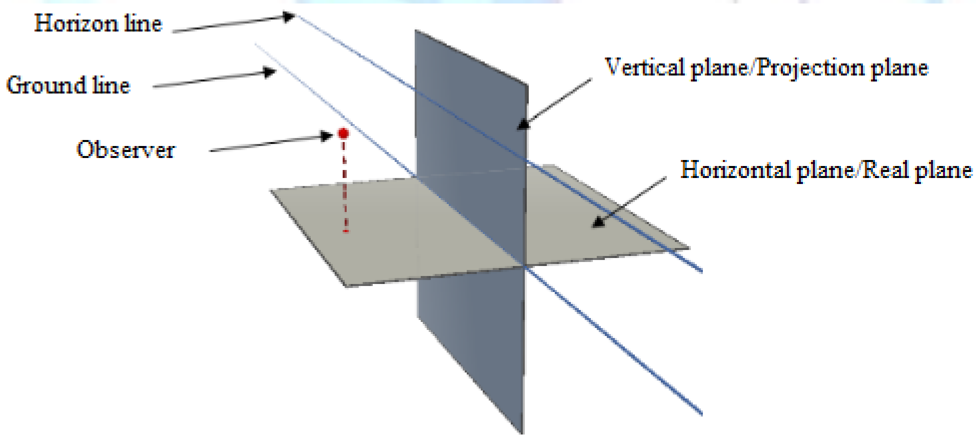
Figure 7 - Horizon Line



Source: Personal Collection

Revisiting the practices of the 15th century with the aid of the software allowed us to identify some of the fundamental elements of perspective representation, which we summarise below. The ground *line* is the intersection of the *vertical plane of projection* with the horizontal plane also called the *real plane.* The points of intersection of the parallel line projections are called *vanishing points*. The straight line that passes through the *vanishing points* is called the *horizon line,* as summarised in Figure 8.

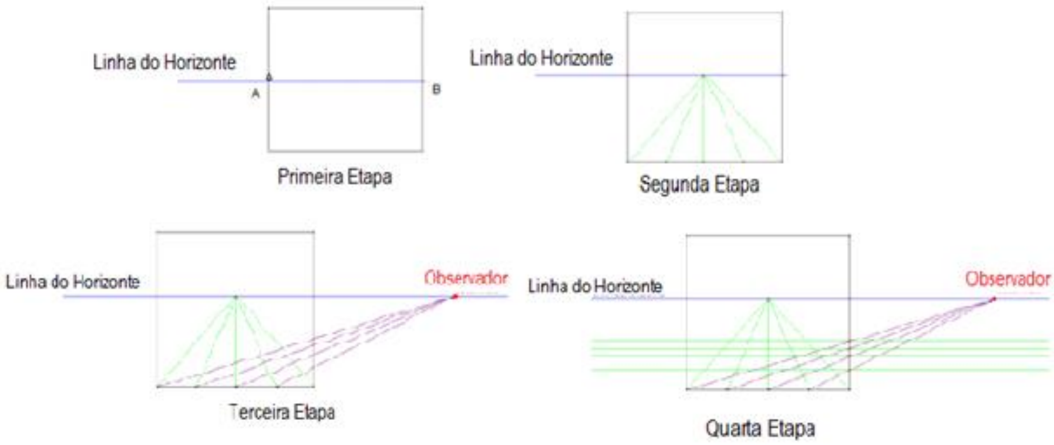
Figure 8 - Elements of the Perspective



Source: Personal Collection

This revisiting also made possible to understand the first rules for the plane representation of three-dimensional figures, which were described as steps by Leon Battista Alberti (1404-1472), in his work "De Pictura", translated into English as "On Painting" (ALBERTI, 1991). Alberti provides a simple and straightforward explanation of these techniques, dividing them into four steps (Figure 9) describing the resources used at the time, and follows those steps to represent a checked floor whose base rests on the *ground line*.

Figure 9 - Alberti's representation rules



Source: Personal Collection

In the first step, the "horizon line" is fixed, in this case$\overleftrightarrow{AB}$, at the height of the observer in relation to the figure to be represented, and parallel to the *ground line*. In the second, the *ground line*, base of the figure, is divided into the number of parts corresponding to the number of pieces of the floor and the midpoint of the segment $\overline{AB}$ of the ground line is the "vanishing point" of the lines that are perpendicular to the plane of the figure. The segments joining the points of the base of the figure to the "vanishing point" (ALBERTI, 1991, p.55) are plotted and will form the projection of the floor lines that are perpendicular to the plane of projection. In the third, the position of the observer is chosen on the horizon line, "vanishing point" of the straight lines of the floor that are parallel, and line segments are drawn between the observer and the points of the base of the figure, on the ground line. The intersections of these segments with the lateral line of the picture canvas serve to trace the lines parallel to the plane of the figure and give the notion of depth of the floor pieces on the image. In the fourth step, the lines parallel to the horizon line are drawn, defining the floor of the image. With this technique, Alberti was able to show the painters of the time how to create the image of a floor in perspective and with vanishing points. We also used Alberti’s steps to guide the introduction to perspective techniques in our activities.

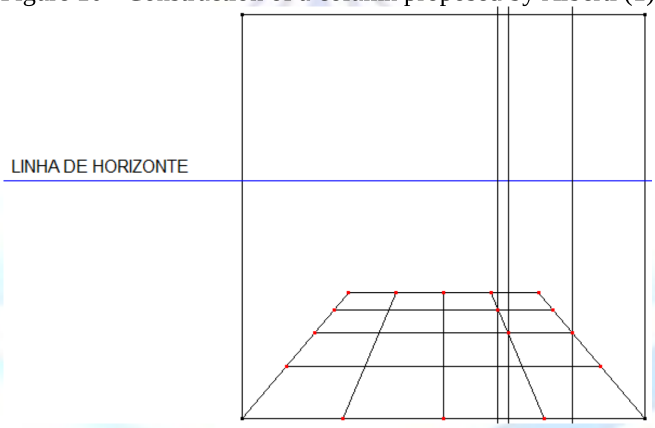
Alberti points out in his book that the method should be understood at once, as the process is very simple:

> I briefly outlined the general context and, I believe, in a not altogether obscure way, but I realise that the content is such that, as I cannot demand praise for eloquence in

> clarifying it, the reader who does not understand it at the first contact will probably never understand it, no matter how hard they try. (ALBERTI, 1435, On Painting (1991), p. 58-59)

It then explains how to create objects that have the front face parallel to the plane of the canvas, using the previously created floor, the vanishing point used to create the floor and vertical lines parallel to the side edges of the canvas (Figure 10).

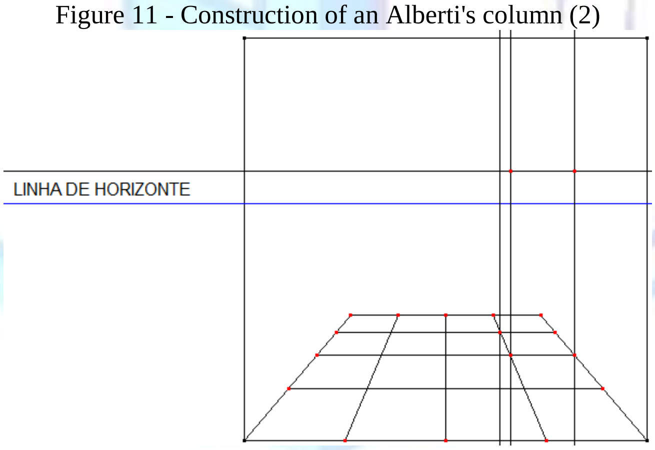
Figure 10 - Construction of a column proposed by Alberti (1)



Source: Personal Collection

Once these lines are drawn, the height of the object to be drawn with a line parallel to the ground line is defined (Figure 11).

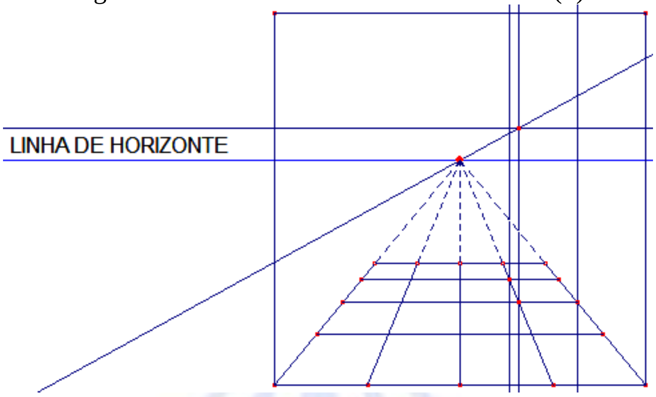
Figure 11 - Construction of an Alberti's column (2)



Source: Personal Collection

A straight line is drawn from the vanishing point, passing through the point of intersection of the vertical lines and the line parallel to the ground line, forming the upper part of the object (Figure 12).

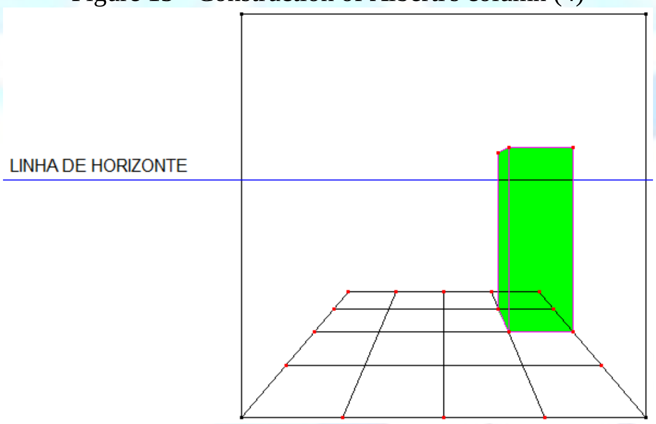
Figure 12 - Construction of Alberti's column (3)



Source: Personal Collection

Finally, the desired object is drawn from the intersections of the straight lines, as shown in Figure 13.

Figure 13 - Construction of Alberti's column (4)



Source: Personal Collection

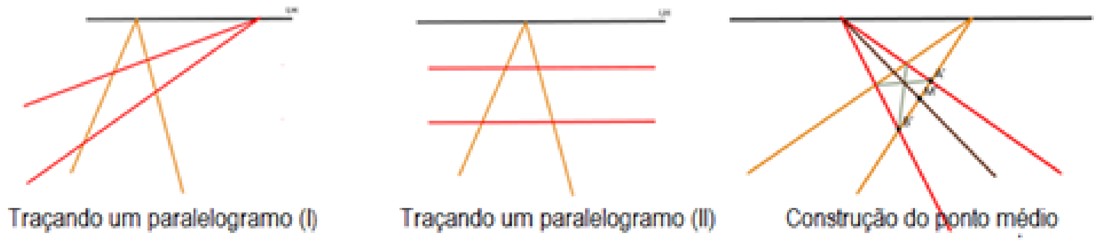
With these steps, Alberti brought to the world of painting a set of techniques that would revolutionise the concept of art. We used Alberti's concepts to show students the evolution of projection and perspective techniques, from the instruments used for painting with depth notion to the geometric concepts that arose with Brunelleschi and were perfected by Alberti. Through the knowledge and techniques expressed by Alberti, we were able to discuss and reconstruct, with the help of the software, the elements involved in the steps detailed in the book, to show students how this creation of a projected environment works in practice, and to explore the geometric properties underlying these steps.

Throughout the activities of the first meeting, the discussions allowed establishing the relationships between the "pole of the seen and the pole of the known" in the plane representations obtained through the basic constructions executed. It also allowed to clarify the coding rules established in the introductory representations in perspective.

Data analysis

The presentation of historical aspects related to perspective allowed us to draw a parallel between what is seen and what is known, to sharpen participants' perceptions and to verify if the poles of the "seen" and the "known" were in harmony; we could also observe if they established the relationship figure-perception through the perception of spatial relationships and positioning, as proposed by Gutiérrez. Few have been able to use the elements of the images to perceive the notion of depth and we have found that this perception is easier if the image is constructed based on the conical perspective. At the end of the meeting, we presented the projection and perspective techniques using the "Horizon Line" and the "Vanishing Point" as a foundation for our constructions. We showed how to construct the projection of a parallelogram in two situations: I. The two pairs of opposite sides are not parallel to the ground line (Figure 4 - I); II. One pair is parallel to the ground line (Figure 4-II).

Figure 14 - Tracing a parallelogram and the midpoint of the diagonals



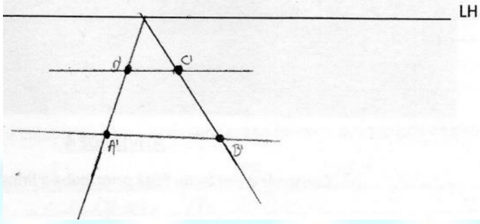
Source: Personal Collection

We discuss how to find, in the representation with pencil and ruler, the point corresponding to the midpoint of one side of the parallelogram (Figure 4, right), showing that the line that joins the midpoints of opposite sides of a parallelogram is parallel to the other two sides and passes through the meeting point of the diagonals, which intersect at the midpoints. Still in the second meeting, we proposed activities to work with those techniques, as shown below.

ACTIVITY 1. A parallelogram *ABCD* is situated in the horizontal plane. Given the horizon line (*LH*), construct its perspective on the vertical plane, corresponding to the sheet of paper; name *A ', B', C 'and D'* the perspectives of points *A, B, C and D,* respectively.

We tried to evaluate whether the techniques of plotting parallels and parallelograms were enough for the participants to reproduce and solve the problem proposed, based on one of Parzysz's hypotheses, that the knowledge of techniques and methods is of fundamental importance to learn the representations of the figures. All participants were successful in reproducing the constructions of the parallelograms; some of them constructed the two types of parallelograms seen during the discussion. We note that the simplest techniques are easily assimilated and reproduced, allowing variations, without loss of information, as seen in Figure 15.

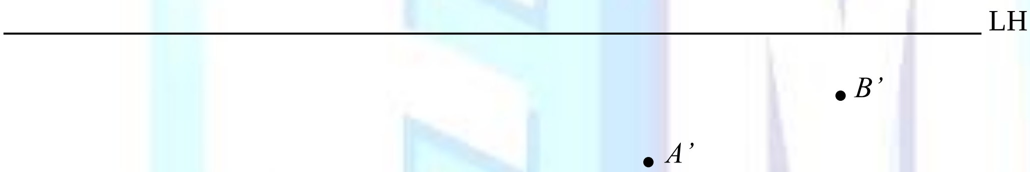
Figure 15 - Response of participant A10 to Activity 1



Source: Personal Collection

In activities 1 to 3, we tried to verify the interference that the conceptual and technical difficulties described by Gutiérrez had in the unusual constructions.

ACTIVITY 2. A' and B' are two points in a photo. Construct the midpoint of the segment $\overline{A'B'}$, being given the horizon line.



We noticed that some had forgotten the construction steps and only built the parallelogram with side $\overline{A'B'}$. Six of the ten participants did not use the diagonals intersection to determine the midpoint of the sides and some answers show that they had not assimilated the technique, because they used the concept of midpoint of the plane geometry, i.e., pole of the "known" in the visual representation, pole of the "seen"; four of the participants completed the construction of the midpoint.

In the responses of A3 (Figure 16) and A10 (Figure 17) we have, respectively, a construction that follows the procedures we presented, with an addition to the "known" pole and one that rests on the "seen" pole, by marking the midpoint of $\overline{A'B'}$. The resumption of the same ideas should support the resolution of the activity 3.

Figure 16 – Response of A3 to Activity 2

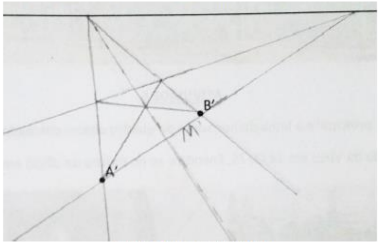
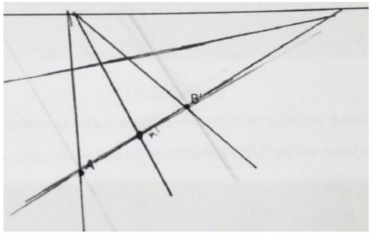
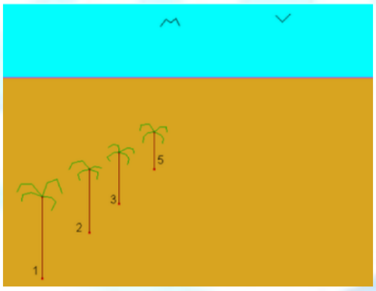


Figure 17 – Response of A10 to Activity 2



Source: Personal Collection

ACTIVITY 3. The trees in the photo below are equidistant. Build the missing tree between 3 and 5.



In the image, the trees act as fixed points and the horizon line is the straight line that divides the desert sky, in which one must mark the vanishing point of parallel lines. It is important to emphasise that the participants had to think about how they would use one of the techniques seen, from what they already knew about midpoints and analysis of patterns. We verified that four of them presented the solution with the correct and complete use of the technique in question, five solved using partially the technique and one did not reach the solution. The statement caused additional difficulty, because in addition to not requesting the use of the technique directly, the Horizon Line is implicit.

The responses of A7 (Figure 18) and A1 (Figure 19) show the difference between understanding the use of a technique and understanding it partially, failing to execute it.

Figure 18 - Response of A7 to Activity 3

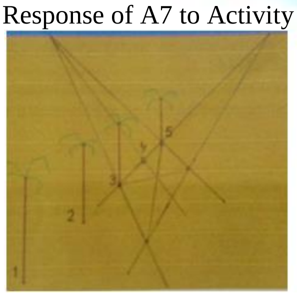
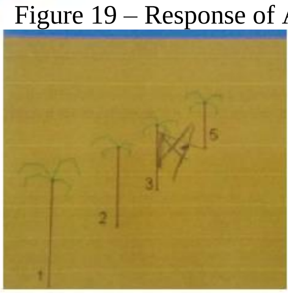


Figure 19 – Response of A1 to Activity 3



Source: Personal Collection

In the third meeting, we presented and discussed constructions with 3D Dynamic Geometry software, in which we represented the projection planes and projections of horizontal plane elements (plane and three-dimensional figures) highlighting the location of the real objects and the projected image, of the horizon lines and of the observer. We showed the projection of a square and a cube situated in the real plane (Figures 10 and 11), whose constructions take place through the observer's semi-lines at the vertices of the figures and intersect the projection plane, marking the vertices of the projected images.

Figure 20 - Projection of a square

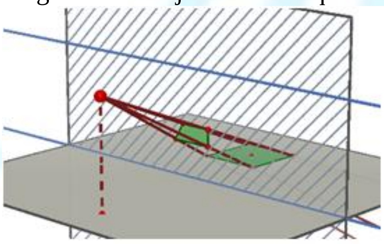
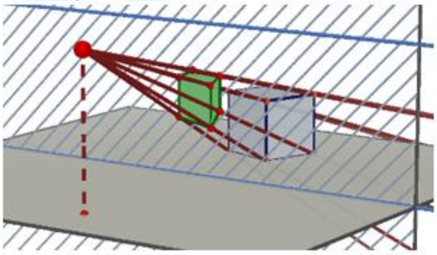


Figure 21 – Projection of a cube



Source: Personal Collection

In Figure 21, we have a plane representation of a three-dimensional figure, from the projection of the vertices of the cube "visible" to the observer.

We concluded the discussion of the use of software in the third meeting, exploring the question of determining the position of the observer, given the image of a parallelogram; then the participants answered a final questionnaire about the activities developed.

The use of the software made it possible to interact with the projection environment dynamically and to show how the position of the observer changes the height of the horizon line, which made the understanding and behaviour of the elements more feasible. The participants stated that the use of the software aroused interest in the subject, not only because of the possibility to visualise what was discussed in the previous meetings, but also to work with those tools, in the case of reconstructing images from their projections.

Conclusions

It became evident that the historical study of perspective in works of art can help to understand its importance in the representations and that the use of software to understand and use the techniques in other situations positively assists in the interest for geometry; besides, we reached one of the objectives, which was to introduce elements of the projective geometry through the initial ideas of the perspective and with the support of the geometry of position.

In accordance with Souza (2010) and Bedim (2011), we conclude that the triad - teacher, material, student - is essential to understand the representations, since the "known" of the student may not be enough to understand the various representations. Just like Miranda (2006), Kodama (2006) and Meneguzzi (2009), we were able to reconstruct the models of interest to explore properties of the perspective and to show to the participants how this can be done with paper and pencil or with a dynamic geometry software. We also linked the models of representation studied by Gutiérrez (1998) to the poles of the "seen" and the "known", defined by Parzysz (1998).

The activities of the second meeting gave us some of the elements to answer the second research question: "Does the use of software to work with projections of figures make easier for the students to understanding topics of projective geometry?". We presented again the techniques of representation of straight lines and polygons in a vertical plane of projection so that, in the activity, with the software, we could verify its contribution to the student's apprehension of aspects of the projective geometry. The participants were able to understand more clearly how the representations function and how their constructs would be represented in a three-dimensional environment. This meeting was fundamental to the research, since it showed that the use of the software aroused interest in content and representations. It became clear that Parzysz (1988) mentions the loss of information in plane representations of spatial figures because, when we eliminate this loss with the use of software, what we see (the "seen") equals what we know (the "known") and there is a gain in the understanding, since the software allows both representations to be visualised and there is no longer the difficulty of having to mentally associate the two concepts, the "seen" and the "known".

We emphasise that a more in-depth research on the use of software of dynamic geometry with situations in which the participants are a very active part of the constructions is necessary. The full use of the geometric tools of the software makes possible to apply what is known, without depending on drawing skills, and to improve one's visualisation capacity. We believe that situations of this kind can increase students' interest in geometry, regardless of their level of education.

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